University of Arkansas
Fayetteville, Arkansas

The Four Color Theorem and Branched Covers of S^2

Yo'av Rieck (University of Arkansas)

Yasushi Yamashita (Nara Women's University)

November 28, 2005, Osaka Sangyo Daigaku

大阪産業大学 教養部





The four Color Problem

In 1852 Francis Guthrie colored a map of England using only four colors so that neighboring counties are colored distinct colors, and conjectured:

The four Color Problem

In 1852 Francis Guthrie colored a map of England using only four colors so that neighboring counties are colored distinct colors, and conjectured:

Conjecture 1 (The four Color Problem) *Any map can be four colored.*

The four Color Problem

In 1852 Francis Guthrie colored a map of England using only four colors so that neighboring counties are colored distinct colors, and conjectured:

Conjecture 1 (The four Color Problem) *Any map can be four colored.*

In this conjecture we assume that the countries are on earth and each country is connected.

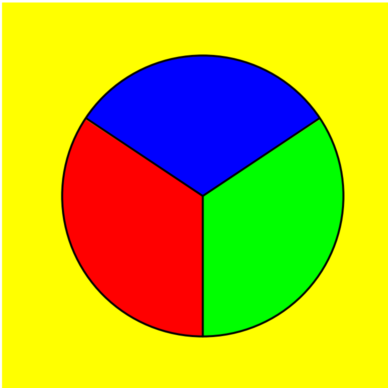
The four Color Problem

In 1852 Francis Guthrie colored a map of England using only four colors so that neighboring counties are colored distinct colors, and conjectured:

Conjecture 1 (The four Color Problem) *Any map can be four colored.*

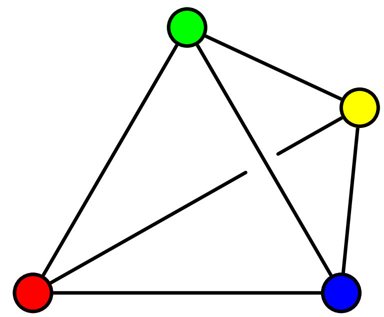
In this conjecture we assume that the countries are on earth and each country is connected.

Note that the map M_{stan} below cannot be three colored:

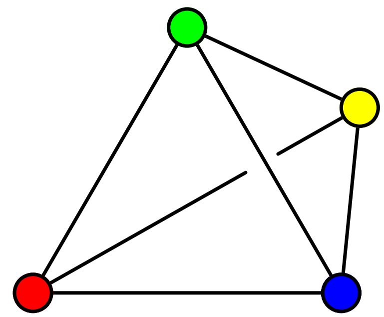


Let M be a map. We begin by dualizing the map: we put a vertex in each country and connect two vertices by an edge if and only if they correspond to neighboring countries. The dual of M_{stan} is K_4. We call that the standard triangulation of S^2, denotes $\mathcal{T}_{\text{stan}}$:

Let M be a map. We begin by dualizing the map: we put a vertex in each country and connect two vertices by an edge if and only if they correspond to neighboring countries. The dual of M_{stan} is K_4. We call that the standard triangulation of S^2, denotes $\mathcal{T}_{\text{stan}}$:

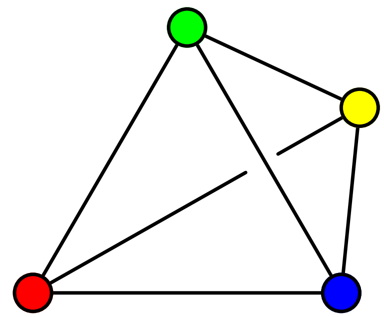


Let M be a map. We begin by dualizing the map: we put a vertex in each country and connect two vertices by an edge if and only if they correspond to neighboring countries. The dual of M_{stan} is K_4. We call that the standard triangulation of S^2, denotes $\mathcal{T}_{\text{stan}}$:



By adding edges to the dual we may assume the dual is a triangulation. Thus the Four Color Problem can be rephrased as:

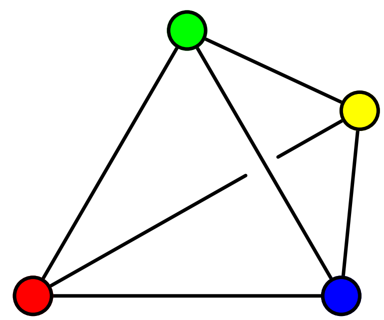
Let M be a map. We begin by dualizing the map: we put a vertex in each country and connect two vertices by an edge if and only if they correspond to neighboring countries. The dual of M_{stan} is K_4. We call that the standard triangulation of S^2, denotes $\mathcal{T}_{\text{stan}}$:



By adding edges to the dual we may assume the dual is a triangulation. Thus the Four Color Problem can be rephrased as:

Conjecture 2 *The vertices of a triangulation of S^2 can be four colored.*

Let M be a map. We begin by dualizing the map: we put a vertex in each country and connect two vertices by an edge if and only if they correspond to neighboring countries. The dual of M_{stan} is K_4. We call that the standard triangulation of S^2, denotes $\mathcal{T}_{\text{stan}}$:



By adding edges to the dual we may assume the dual is a triangulation. Thus the Four Color Problem can be rephrased as:

Conjecture 2 *The vertices of a triangulation of S^2 can be four colored.*

These conjectures (now theorem) was proved by Appel and Haken in the 1970's.

Branched Covers

Our first goal is reformulating Conjecture 2 in terms of *branched covers*. For $f : S^2 \to S^2$ we define:

Branched Covers

Our first goal is reformulating Conjecture 2 in terms of *branched covers*. For $f : S^2 \to S^2$ we define:

Definition 3 *f is called a cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^{n}$ and for each j, $f|_{D_j}$ is a homeomorphism.*

Branched Covers

Our first goal is reformulating Conjecture 2 in terms of *branched covers*. For $f : S^2 \to S^2$ we define:

Definition 3 *f is called a cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is a homeomorphism.*

Definition 4 *f is called a branched cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is modelled on $z \to z^n$.*

Branched Covers

Our first goal is reformulating Conjecture 2 in terms of *branched covers*. For $f : S^2 \to S^2$ we define:

Definition 3 *f is called a cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is a homeomorphism.*

Definition 4 *f is called a branched cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is modelled on $z \to z^n$.*

n is called the local degree and the image of the points with local degree greater than 1 is called the branch set, denoted $\mathrm{branch}(f)$. $\mathrm{branch}(f)$ is finite.

Branched Covers

Our first goal is reformulating Conjecture 2 in terms of *branched covers*. For $f : S^2 \to S^2$ we define:

Definition 3 *f is called a cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is a homeomorphism.*

Definition 4 *f is called a branched cover if for any point $p \in S^2$ there is an open disk $D \ni p$ so that $f^{-1}(D)$ is a collection of disks $\{D_j\}_{j=1}^n$ and for each j, $f|_{D_j}$ is modelled on $z \to z^n$.*

n is called the local degree and the image of the points with local degree greater than 1 is called the branch set, denoted $\mathrm{branch}(f)$. $\mathrm{branch}(f)$ is finite.

The maps $z \to z^n$ and their compositions provide examples of branched cover $f : S^2 \to S^2$ with arbitrarily many branch points.

Lifts

Let $f : S^2 \to S^2$ be a branch cover and suppose that $\mathrm{branch}(f) \subset \mathcal{T}_{\mathrm{stan}}^{(0)}$ (the vertices of $\mathcal{T}_{\mathrm{stan}}$). Then we claim:

Lifts

Let $f : S^2 \to S^2$ be a branch cover and suppose that $\mathrm{branch}(f) \subset \mathcal{T}^{(0)}_{\mathrm{stan}}$ (the vertices of $\mathcal{T}_{\mathrm{stan}}$). Then we claim:

Proposition/definition 5 *$f^{-1}(\mathcal{T}_{stan})$ is a triangulation of S^2. A triangulation obtained in this way is called a lift, denoted $\widetilde{\mathcal{T}}$.*

Lifts

Let $f : S^2 \to S^2$ be a branch cover and suppose that $\text{branch}(f) \subset \mathcal{T}_{\text{stan}}^{(0)}$ (the vertices of $\mathcal{T}_{\text{stan}}$). Then we claim:

Proposition/definition 5 $f^{-1}(\mathcal{T}_{stan})$ *is a triangulation of* S^2. *A triangulation obtained in this way is called a lift, denoted* $\widetilde{\mathcal{T}}$.

The degree of a vertex v, $\deg(v)$, is the number of edges adjacent to v.

Lifts

Let $f : S^2 \to S^2$ be a branch cover and suppose that $\mathrm{branch}(f) \subset \mathcal{T}^{(0)}_{\mathrm{stan}}$ (the vertices of $\mathcal{T}_{\mathrm{stan}}$). Then we claim:

Proposition/definition 5 *$f^{-1}(\mathcal{T}_{stan})$ is a triangulation of S^2. A triangulation obtained in this way is called a lift, denoted $\widetilde{\mathcal{T}}$.*

The degree of a vertex v, $\deg(v)$, is the number of edges adjacent to v.

Proposition 6 *If $\widetilde{\mathcal{T}}$ is a lift then:*

1. *The vertices of $\widetilde{\mathcal{T}}$ are four colorable.*
2. *The degree of every vertex of $\widetilde{\mathcal{T}}$ is divisible by 3.*

Lifts

Let $f : S^2 \to S^2$ be a branch cover and suppose that $\mathrm{branch}(f) \subset \mathcal{T}^{(0)}_{\mathrm{stan}}$ (the vertices of $\mathcal{T}_{\mathrm{stan}}$). Then we claim:

Proposition/definition 5 $f^{-1}(\mathcal{T}_{stan})$ *is a triangulation of* S^2*. A triangulation obtained in this way is called a lift, denoted* $\widetilde{\mathcal{T}}$*.*

The degree of a vertex v, $\deg(v)$, is the number of edges adjacent to v.

Proposition 6 *If* $\widetilde{\mathcal{T}}$ *is a lift then:*

1. *The vertices of* $\widetilde{\mathcal{T}}$ *are four colorable.*
2. *The degree of every vertex of* $\widetilde{\mathcal{T}}$ *is divisible by 3.*

Naïvely we might ask if these statements are "if and only if" statements. (No!)

We explain Part (2) of Proposition 6 first. $p \in S^2$ is a vertex of $\widetilde{\mathcal{T}}$ if and only if p maps to a vertex $\mathcal{T}_{\text{stan}}$. The vertices of $\mathcal{T}_{\text{stan}}$ have valence 3. If the local degree at p is n (possibly $n = 1$) the the valence of p is $3n$.

We explain Part (2) of Proposition 6 first. $p \in S^2$ is a vertex of $\widetilde{\mathcal{T}}$ if and only if p maps to a vertex $\mathcal{T}_{\text{stan}}$. The vertices of $\mathcal{T}_{\text{stan}}$ have valence 3. If the local degree at p is n (possibly $n = 1$) the the valence of p is $3n$.

Part (1) is easy: we can four color the vertices of $\mathcal{T}_{\text{stan}}$ and lift the colors.

We explain Part (2) of Proposition 6 first. $p \in S^2$ is a vertex of $\widetilde{\mathcal{T}}$ if and only if p maps to a vertex $\mathcal{T}_{\mathrm{stan}}$. The vertices of $\mathcal{T}_{\mathrm{stan}}$ have valence 3. If the local degree at p is n (possibly $n = 1$) the the valence of p is $3n$.

Part (1) is easy: we can four color the vertices of $\mathcal{T}_{\mathrm{stan}}$ and lift the colors.

Notice that the coloring is given in a particularly simple form: say the colors are g, r, b, y, suppose the color at a vertex p is g. Then around p the colors are r, b, y, r, b, y, r, b, y, r,... ordered cyclically.

We explain Part (2) of Proposition 6 first. $p \in S^2$ is a vertex of $\widetilde{\mathcal{T}}$ if and only if p maps to a vertex $\mathcal{T}_{\text{stan}}$. The vertices of $\mathcal{T}_{\text{stan}}$ have valence 3. If the local degree at p is n (possibly $n = 1$) the the valence of p is $3n$.

Part (1) is easy: we can four color the vertices of $\mathcal{T}_{\text{stan}}$ and lift the colors.

Notice that the coloring is given in a particularly simple form: say the colors are g, r, b, y, suppose the color at a vertex p is g. Then around p the colors are r, b, y, r, b, y, r, b, y, r,... ordered cyclically.

Thus four coloring a lift is easy.

Main Theorems

Definition 7 *We say that a triangulations* $\mathcal{T}$ *embeds in* $\widetilde{\mathcal{T}}$ *(denoted* $\mathcal{T} \subset \widetilde{\mathcal{T}}$*) if the vertices (resp. edges) of* $\mathcal{T}$ *are a subset of the vertices (resp. edges) of* $\widetilde{\mathcal{T}}$*.*

Main Theorems

Definition 7 *We say that a triangulations* $\mathcal{T}$ *embeds in* $\widetilde{\mathcal{T}}$ *(denoted* $\mathcal{T} \subset \widetilde{\mathcal{T}}$*) if the vertices (resp. edges) of* $\mathcal{T}$ *are a subset of the vertices (resp. edges) of* $\widetilde{\mathcal{T}}$*.*

Theorem 8 *The vertices of a triangulation* $\mathcal{T}$ *are four-colorable if and only if* $\mathcal{T}$ *embeds in a lift* $\widetilde{\mathcal{T}}$*.*

Main Theorems

Definition 7 *We say that a triangulations* $\mathcal{T}$ *embeds in* $\widetilde{\mathcal{T}}$ *(denoted* $\mathcal{T} \subset \widetilde{\mathcal{T}}$*) if the vertices (resp. edges) of* $\mathcal{T}$ *are a subset of the vertices (resp. edges) of* $\widetilde{\mathcal{T}}$*.*

Theorem 8 *The vertices of a triangulation* $\mathcal{T}$ *are four-colorable if and only if* $\mathcal{T}$ *embeds in a lift* $\widetilde{\mathcal{T}}$*.*

Theorem 9 *A triangulation is a lift if and only if the degree of each vertex is divisible by 3. In that case, we say that the triangulation is 3-divisible.*

Proof of Thm 8: $\mathcal{T}$ embeds in a lift implies $\mathcal{T}$ is four colorable: we (more-or-less) showed this in Proposition 6.

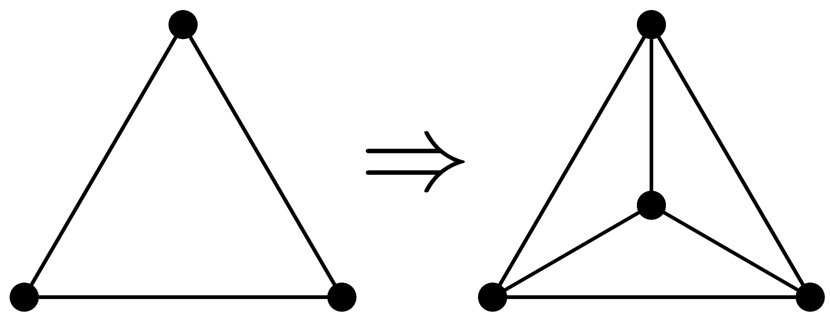
Proof of Thm 8: $\mathcal{T}$ embeds in a lift implies $\mathcal{T}$ is four colorable: we (more-or-less) showed this in Proposition 6.

$\mathcal{T}$ is four colorable implies $\mathcal{T}$ embeds in a lift: suppose that the vertices of $\mathcal{T}$ are four colorable. Any four coloring defines a map to $(S^2, \mathcal{T}_{\text{stan}})$ by sending each triangle to the triangle of $\mathcal{T}_{\text{stan}}$ with the same colors. Endow the two copies of S^2 with an orientation. If f restricted to a triangle F is orientation reversing, subdivide F into 3 triangles. This will ensure the map is orientation preserving on all triangles. It is now straight forward to see that the map is a branched cover; thus we embedded $\mathcal{T}$ in a lift.

Proof of Thm 8: $\mathcal{T}$ embeds in a lift implies $\mathcal{T}$ is four colorable: we (more-or-less) showed this in Proposition 6.

$\mathcal{T}$ is four colorable implies $\mathcal{T}$ embeds in a lift: suppose that the vertices of $\mathcal{T}$ are four colorable. Any four coloring defines a map to $(S^2, \mathcal{T}_{\text{stan}})$ by sending each triangle to the triangle of $\mathcal{T}_{\text{stan}}$ with the same colors. Endow the two copies of S^2 with an orientation. If f restricted to a triangle F is orientation reversing, subdivide F into 3 triangles. This will ensure the map is orientation preserving on all triangles. It is now straight forward to see that the map is a branched cover; thus we embedded $\mathcal{T}$ in a lift.

Note: the only move necessary is:



Proof of Thm 9: lift implies 3-divisible: this was established in Proposition 6.

Proof of Thm 9: lift implies 3-divisible: this was established in Proposition 6.

3-divisible implies lift: assume $\mathcal{T}$ is 3-divisible. Pick one triangle of $\mathcal{T}$ and map it to some triangle of $\mathcal{T}_{\text{stan}}$. Let v be a vertex of T and (imitating the process of analytic continuation) we map the triangles around v to $(S^2, \mathcal{T}_{\text{stan}})$; this is possible since $3 | \deg v$. A map defined in that way is a branched cover.

Proof of Thm 9: lift implies 3-divisible: this was established in Proposition 6.

3-divisible implies lift: assume $\mathcal{T}$ is 3-divisible. Pick one triangle of $\mathcal{T}$ and map it to some triangle of $\mathcal{T}_{\text{stan}}$. Let v be a vertex of T and (imitating the process of analytic continuation) we map the triangles around v to $(S^2, \mathcal{T}_{\text{stan}})$; this is possible since $3|\deg v$. A map defined in that way is a branched cover.

It is not clear that f is well-defined. To show that f is well defined we use a monodromy argument. Assume there exists a cycle c along which f is ill-defined. We may assume c is embedded. We then show that after removing one triangle from D, f is still inconsistent along the boundary. Thus we finally arrive at a disk of area zero,which is absurd.

An aside: the color of edges

It is well-known that four coloring vertices of a triangulation is equivalent to thee-coloring its edges in such way that around each triangle all three colors appear. But what are these colors? Let $\mathcal{T}$ be a four-colored triangulation and let $f : (S^2, \mathcal{T}) \to (S^2, \mathcal{T}_{\text{stan}})$ be the map constructed above.

An aside: the color of edges

It is well-known that four coloring vertices of a triangulation is equivalent to thee-coloring its edges in such way that around each triangle all three colors appear. But what are these colors? Let $\mathcal{T}$ be a four-colored triangulation and let $f : (S^2, \mathcal{T}) \to (S^2, \mathcal{T}_{\text{stan}})$ be the map constructed above.

By the Riemann Uniformization Theorem f is conformal. $\widetilde{\mathcal{T}}^{(0)}$ maps to $\mathcal{T}^{(0)}_{\text{stan}}$, and after picking an order on $\mathcal{T}^{(0)}_{\text{stan}}$ we may assume these points are (in order) at 0, 1, ∞, and z. This uniquely defines z.

An aside: the color of edges

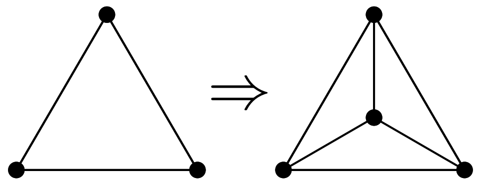
It is well-known that four coloring vertices of a triangulation is equivalent to thee-coloring its edges in such way that around each triangle all three colors appear. But what are these colors? Let $\mathcal{T}$ be a four-colored triangulation and let $f : (S^2, \mathcal{T}) \to (S^2, \mathcal{T}_{\text{stan}})$ be the map constructed above.

By the Riemann Uniformization Theorem f is conformal. $\widetilde{\mathcal{T}}^{(0)}$ maps to $\mathcal{T}_{\text{stan}}^{(0)}$, and after picking an order on $\mathcal{T}_{\text{stan}}^{(0)}$ we may assume these points are (in order) at 0, 1, ∞, and z. This uniquely defines z.

Let T be the ideal tetrahedron with vertices $0, 1, \infty, z$. The six edges of T has 3 dihedral angles (say α, β, and γ) with $\alpha + \beta + \gamma = \pi$. Lifting them gives a 3-coloring of the edges. Thus α, β, and γ are the correct colors.

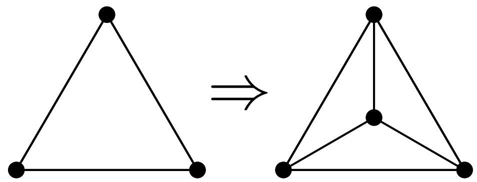
Algorithm

We saw that an embedding in a lift can be obtained using only the move:



Algorithm

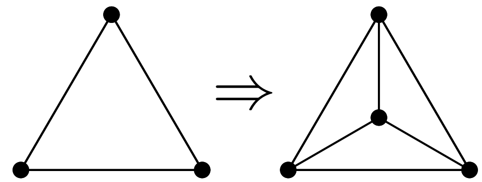
We saw that an embedding in a lift can be obtained using only the move:



- Let M be the incidence matrix (rows = vertices, columns = faces).

Algorithm

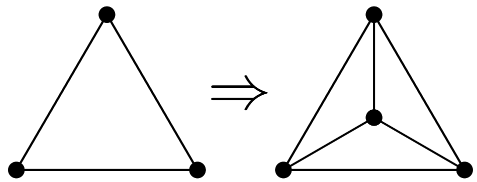
We saw that an embedding in a lift can be obtained using only the move:



- Let M be the incidence matrix (rows = vertices, columns = faces).
- Let $\overrightarrow{b}$ a vector so that b_i is the mod-3 residue of the ith vertex of $\mathcal{T}^{(0)}$.

Algorithm

We saw that an embedding in a lift can be obtained using only the move:

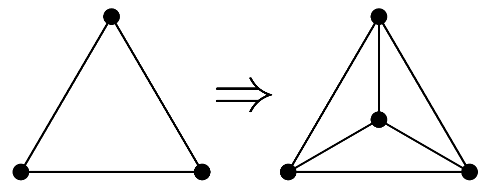


- Let M be the incidence matrix (rows = vertices, columns = faces).
- Let $\overrightarrow{b}$ a vector so that b_i is the mod-3 residue of the ith vertex of $\mathcal{T}^{(0)}$.

Solve: $M\overrightarrow{v} + \overrightarrow{b} = 0 \mod 3$. Then after subdividing the faces for which $v_j = 1$, all the residues are 0. Done.

Algorithm

We saw that an embedding in a lift can be obtained using only the move:



- Let M be the incidence matrix (rows = vertices, columns = faces).
- Let $\overrightarrow{b}$ a vector so that b_i is the mod-3 residue of the ith vertex of $\mathcal{T}^{(0)}$.

Solve: $M\overrightarrow{v} + \overrightarrow{b} = 0 \mod 3$. Then after subdividing the faces for which $v_j = 1$, all the residues are 0. Done.

An example: Haewood's Map

In 1879 Kempe proved the Four Color Theorem; in 1890 Haewood found the mistake in the proof. He constructed a 25-country map as a counterexample to Kempe's *argument* (but not to the theorem!). We ran the algorithm, and the results are given in the handout.

Geometric Genetic Algorithms

Next we consider a bigger map called *Moore's Map*. It has 341 countries and 678 border triangles. In order to look for a solution we use *genetic algorithm*, that is:

Geometric Genetic Algorithms

Next we consider a bigger map called *Moore's Map*. It has 341 countries and 678 border triangles. In order to look for a solution we use *genetic algorithm*, that is:

1. Choose vectors $\{\overrightarrow{v_i}\}_{i=1}^{\text{population}}$.

Geometric Genetic Algorithms

Next we consider a bigger map called *Moore's Map*. It has 341 countries and 678 border triangles. In order to look for a solution we use *genetic algorithm*, that is:

1. Choose vectors $\{\overrightarrow{v_i}\}_{i=1}^{\text{population}}$.
2. For each i, check how many of the conditions $M\overrightarrow{v_i} = \overrightarrow{b}$ hold. This number (between 0 and 341) is called the fitness of v_i.

Geometric Genetic Algorithms

Next we consider a bigger map called *Moore's Map*. It has 341 countries and 678 border triangles. In order to look for a solution we use *genetic algorithm*, that is:

1. Choose vectors $\{\overrightarrow{v_i}\}_{i=1}^{\text{population}}$.
2. For each i, check how many of the conditions $M\overrightarrow{v_i} = \overrightarrow{b}$ hold. This number (between 0 and 341) is called the fitness of v_i.
3. Keep only the fit vectors; use them to generate new vectors by means of crossover (mixing entries of two vectors) and mutation.

Finally, we try to improve the genetic algorithm by using a geometric genetic algorithm:

Finally, we try to improve the genetic algorithm by using a geometric genetic algorithm:

1. We replace the crossover function by a function that keeps the all the entries of one vector and "inject" the entries from another vector arranged on a disk (recall that the coordinates of $\overrightarrow{v_i}$ correspond to faces of $\mathcal{T}$).

Finally, we try to improve the genetic algorithm by using a geometric genetic algorithm:

1. We replace the crossover function by a function that keeps the all the entries of one vector and "inject" the entries from another vector arranged on a disk (recall that the coordinates of $\overrightarrow{v_i}$ correspond to faces of $\mathcal{T}$).
2. We replace the fitness function by a cost function: the sum up the distances between equations [= vertices of $\mathcal{T}$] of $M\overrightarrow{v_i} = \overrightarrow{b}$ that do not hold.

Finally, we try to improve the genetic algorithm by using a geometric genetic algorithm:

1. We replace the crossover function by a function that keeps the all the entries of one vector and "inject" the entries from another vector arranged on a disk (recall that the coordinates of $\overrightarrow{v_i}$ correspond to faces of $\mathcal{T}$).
2. We replace the fitness function by a cost function: the sum up the distances between equations [= vertices of $\mathcal{T}$] of $M\overrightarrow{v_i} = \overrightarrow{b}$ that do not hold.

We ran the generic genetic algorithm for 1,500–2,500 generations with population size 1,000, and got the fitness to be 315–320. The geometric genetic algorithm runs a little slower, so we allowed only 500 generations (same population size). Each run gave fitness of 324–331, and even higher.

Finally, we try to improve the genetic algorithm by using a geometric genetic algorithm:

1. We replace the crossover function by a function that keeps the all the entries of one vector and “inject” the entries from another vector arranged on a disk (recall that the coordinates of $\overrightarrow{v_i}$ correspond to faces of $\mathcal{T}$).
2. We replace the fitness function by a cost function: the sum up the distances between equations [= vertices of $\mathcal{T}$] of $M\overrightarrow{v_i} = \overrightarrow{b}$ that do not hold.

We ran the generic genetic algorithm for 1,500–2,500 generations with population size 1,000, and got the fitness to be 315–320. The geometric genetic algorithm runs a little slower, so we allowed only 500 generations (same population size). Each run gave fitness of 324–331, and even higher.

However, keep in mind that this is still work in progress. We still need to optimize the generic genetic algorithm and the geometric genetic algorithm (find the optimal population size, amount of mutation, etc.).

Yoshihiko Marumoto

Toshio Harikae

Kazuhiro Ichihara

Thank you for organizing a terrific conference for us!