Homology 3-shperes with 2-connected W-graphs

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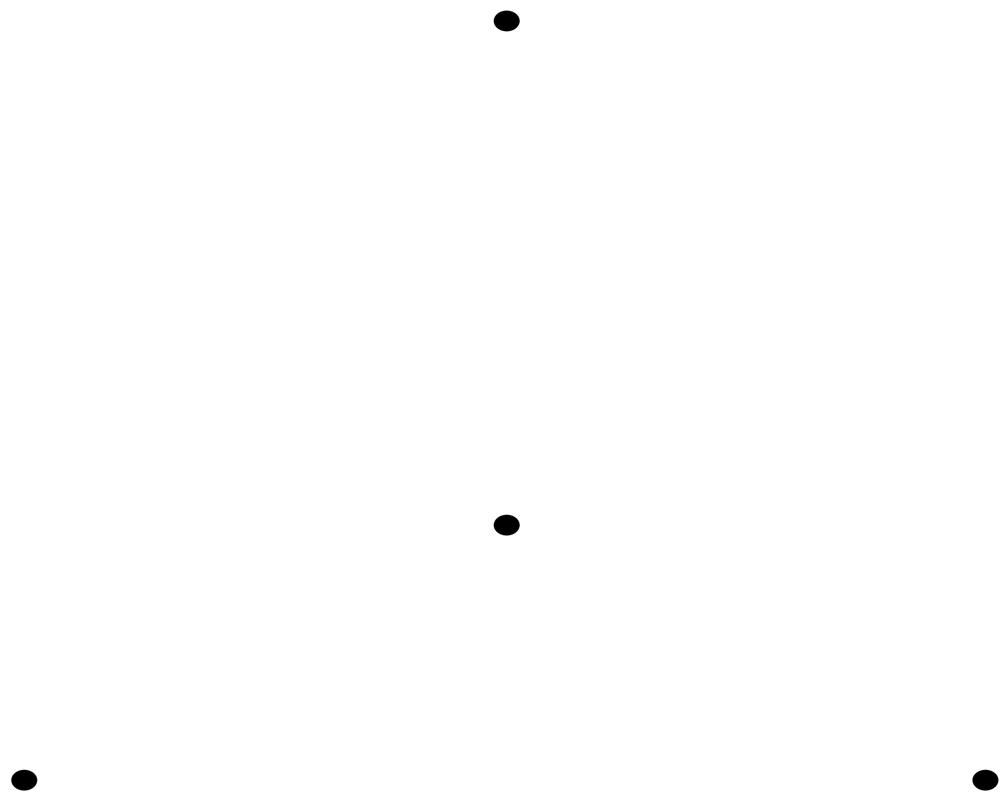
Definitions

- Let D=(F;v,w) be a Heegaard diagram of a closed connected 3-manifold M, where

F is the Heegaard surface such that F=∂H1 ＝ ∂H2, where H1 and H2 are

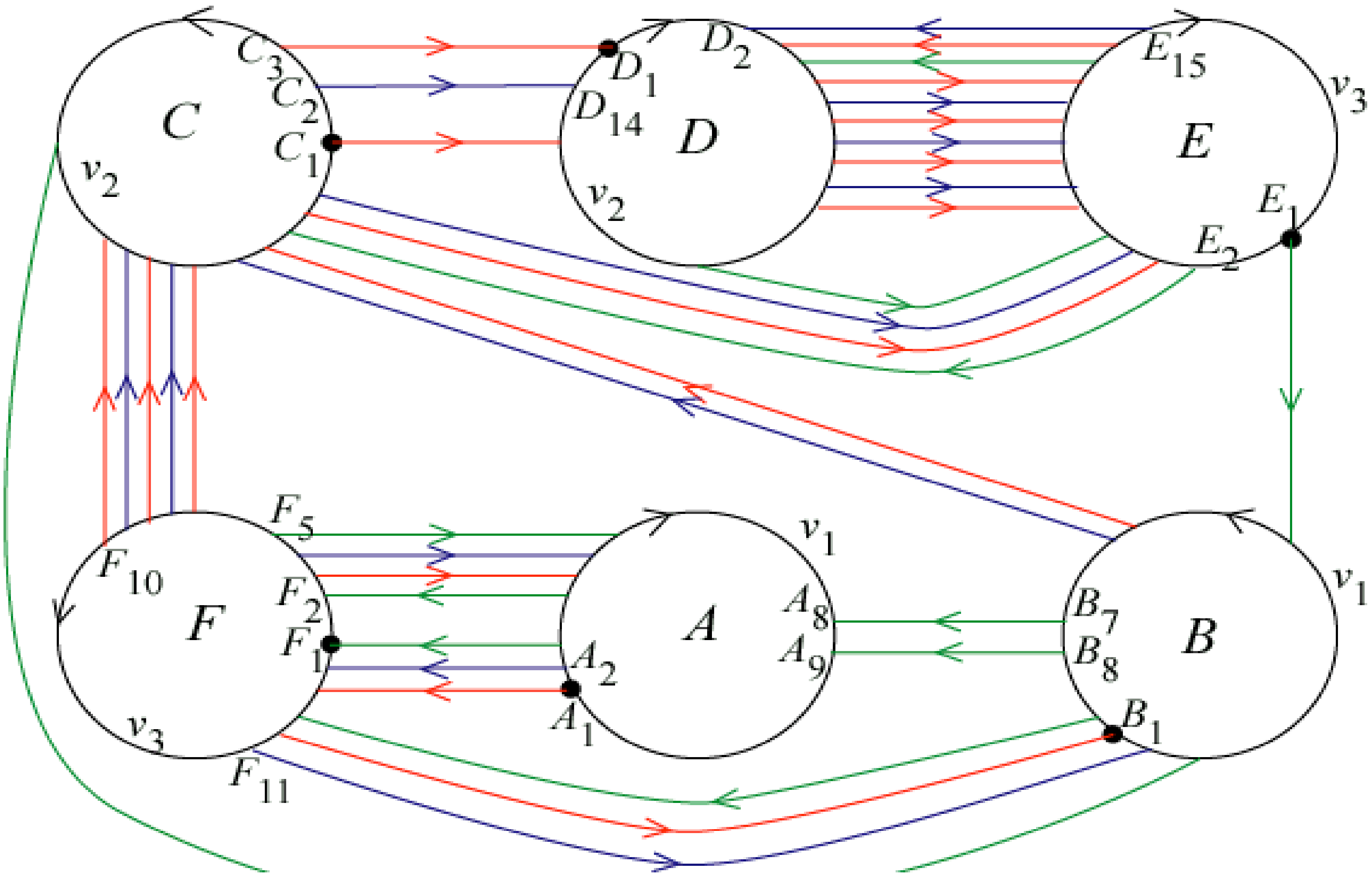
3-dimensional handle, and v,w are complete meridian systems of H1,H2.

A example 1. Let $\pi(M)=\{a,b: a^4BAB=e, b^2ABA=e\}$ be a presentation of the fundamental group induced from D. This gives Poincare homology 3-sphere. Then the following figure is the Whitehead graph which is 3-connected. Note that this diagram is the minimal crossing one.



A group presentation of the standard 3-sphere

- $G=\{a,b,c \mid acb^2CA(BC)^3=1, cb(ac)^2BCA^3=1, acb^3CA(BC)^4=1\}$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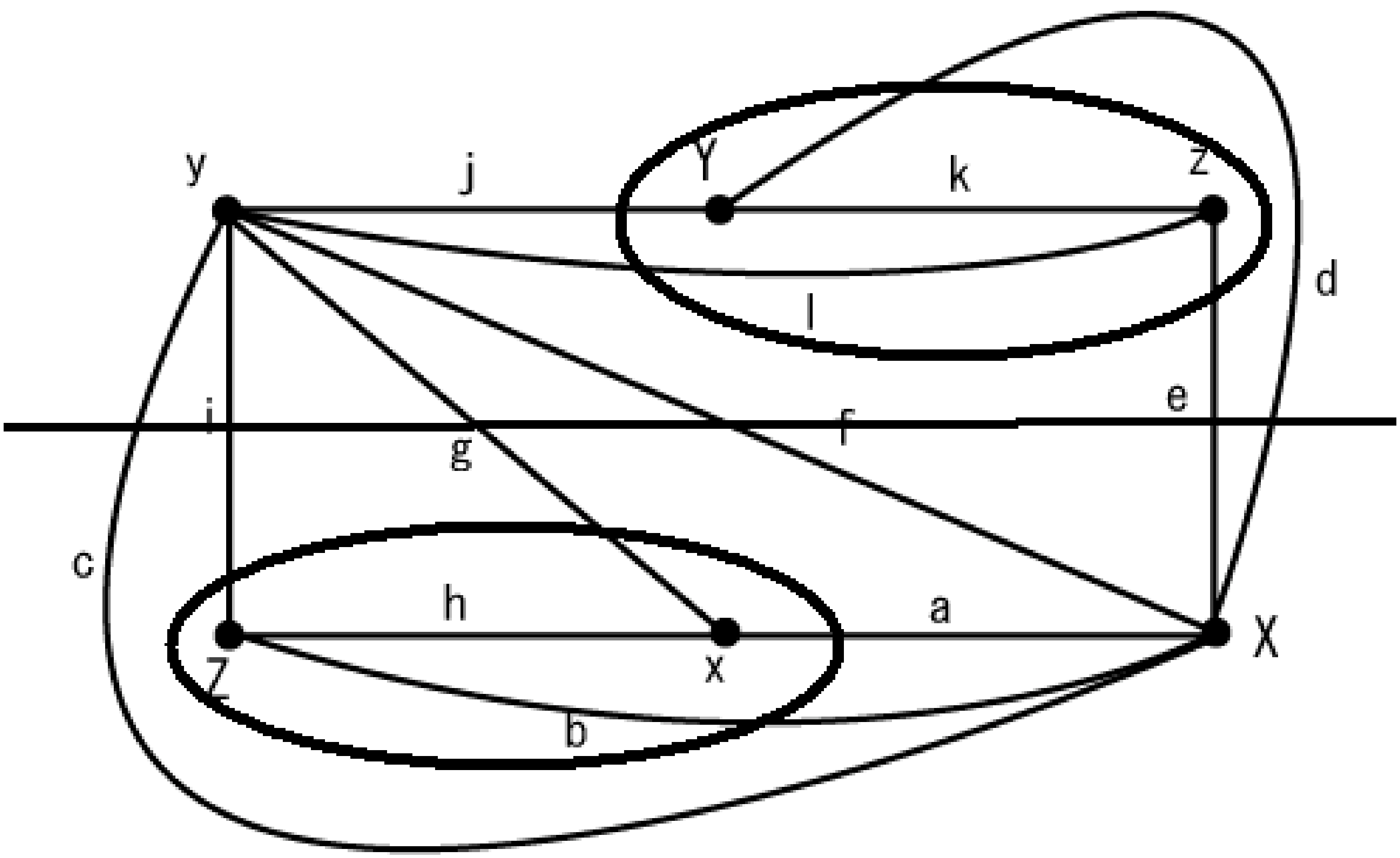
The Whitehead graph (W-graph)



3 equations from identifications

- X:x $a+b+c+d+e+f = a+g+h$
- Y:y $c+I+g+f+l+j = j+k+d$
- Z:z $e+k+l = b+h+I$
- If $g+i > d+e$, reducible by

$g+i+c+f > d+e$.

- If $d+e > g+i$, then also reducible by

$d+e+c+f > g+i$.

If $g+i=d+e$,

If $c+f>0$, then reducible by

$a+b+g+i < a+b+c+f+e+d$

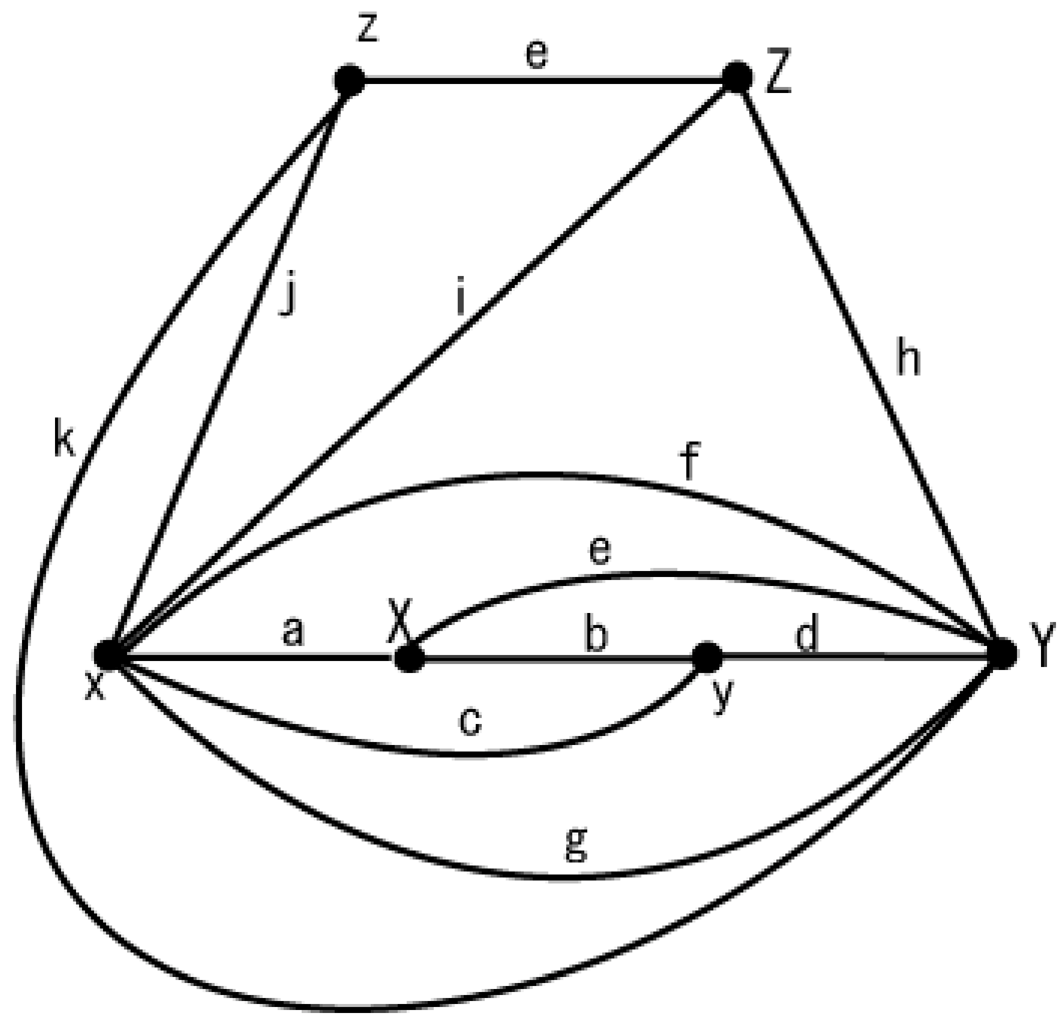
If $c+f=0$, then this meridian (the line) is intersects the meridians v1,v2,v3 in even points.

If $g1+i1 \neq d1+e1$, then $g1+i1 > d1+e1$ or

$g1+i1 < d1+e1$ but this contradicts rexicographic order of the complexity of v1, v2, v3 ($|v1| \leqq |v2| \leqq |v2|$).

By same argument, we have that g2+i2=d2+e2, g3+i2=d2+e2.

But then the manifold is not a homology 3-sphere, that is, all meridians v1,v2,v3 have even intersections with the line (a meridian).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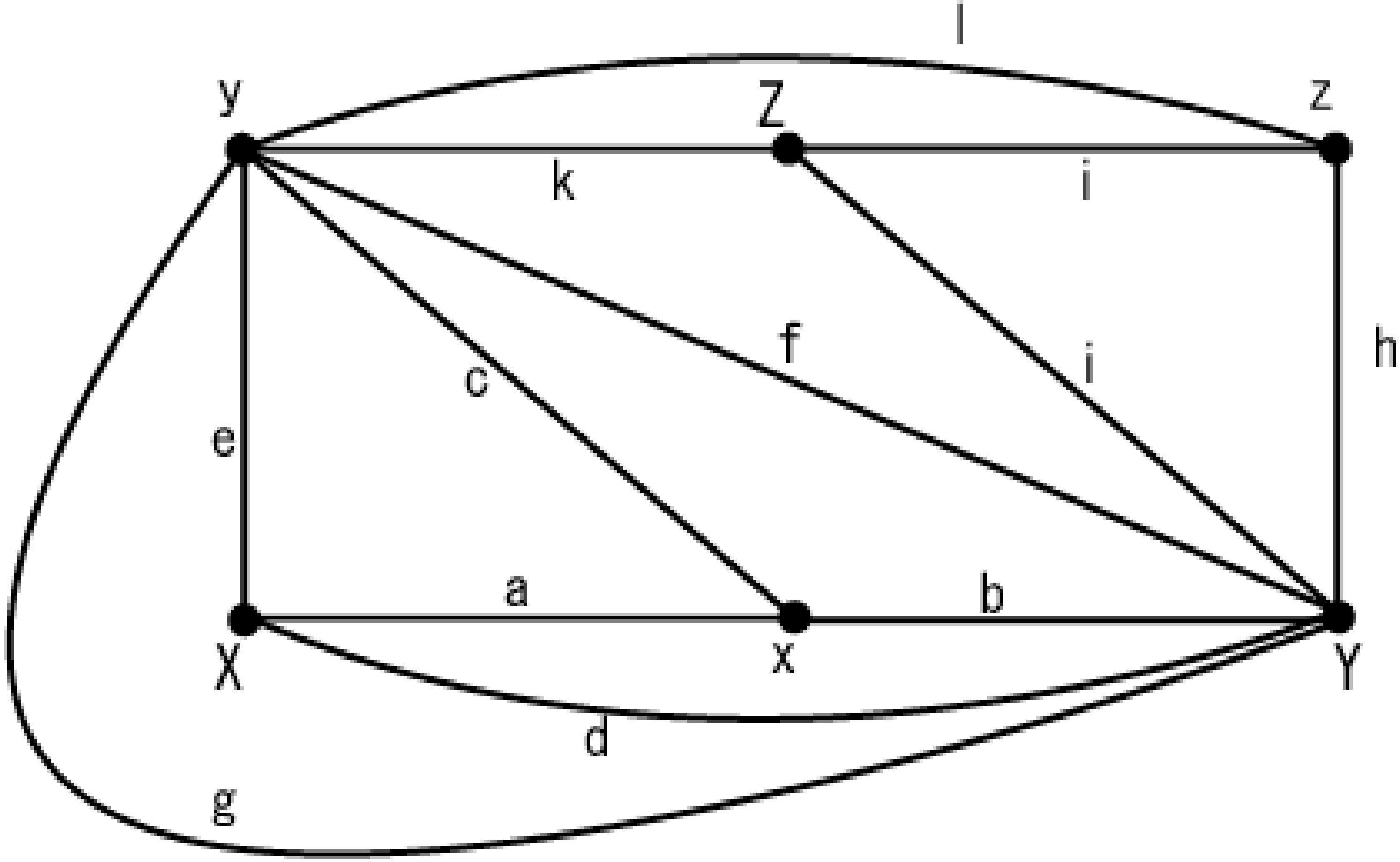

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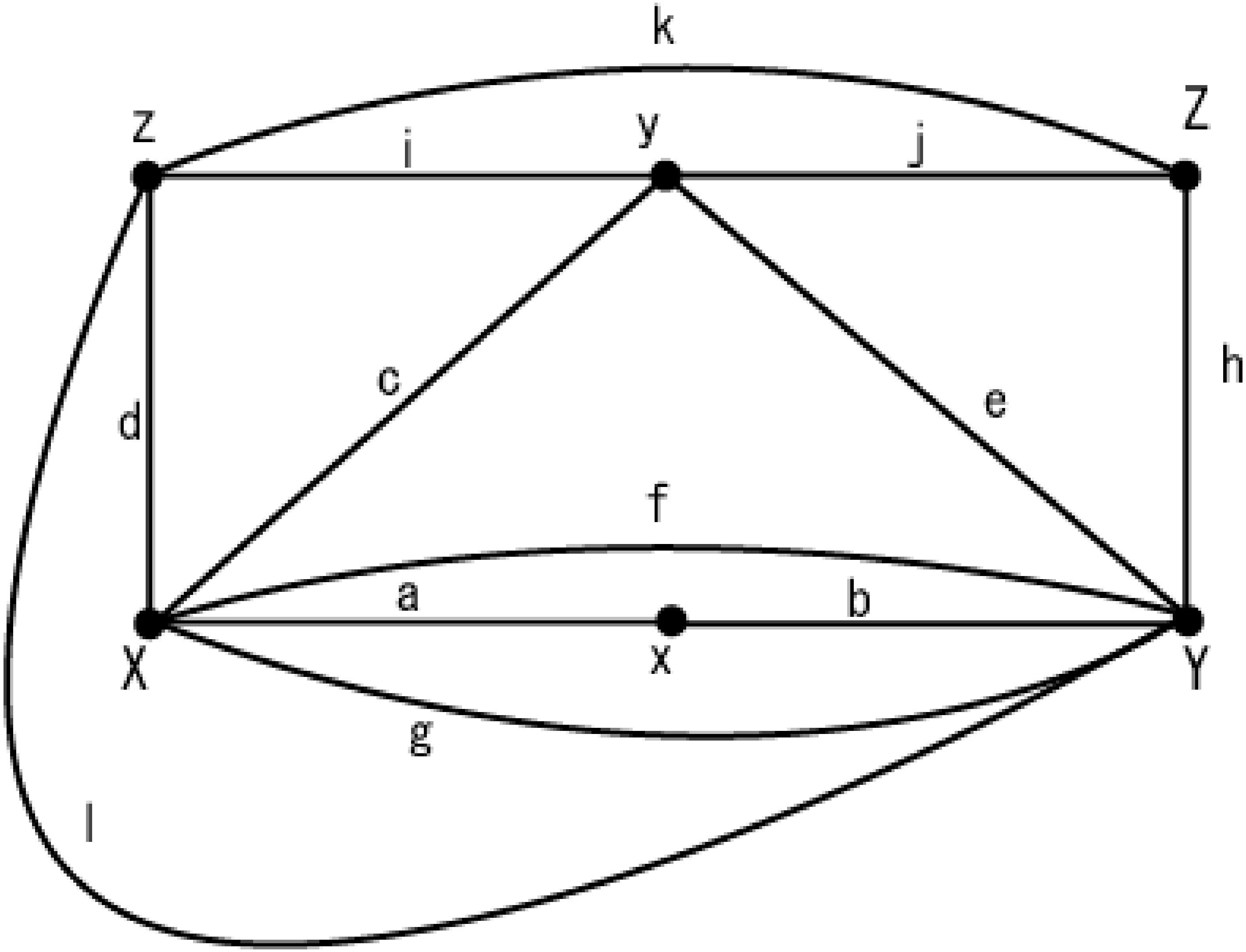
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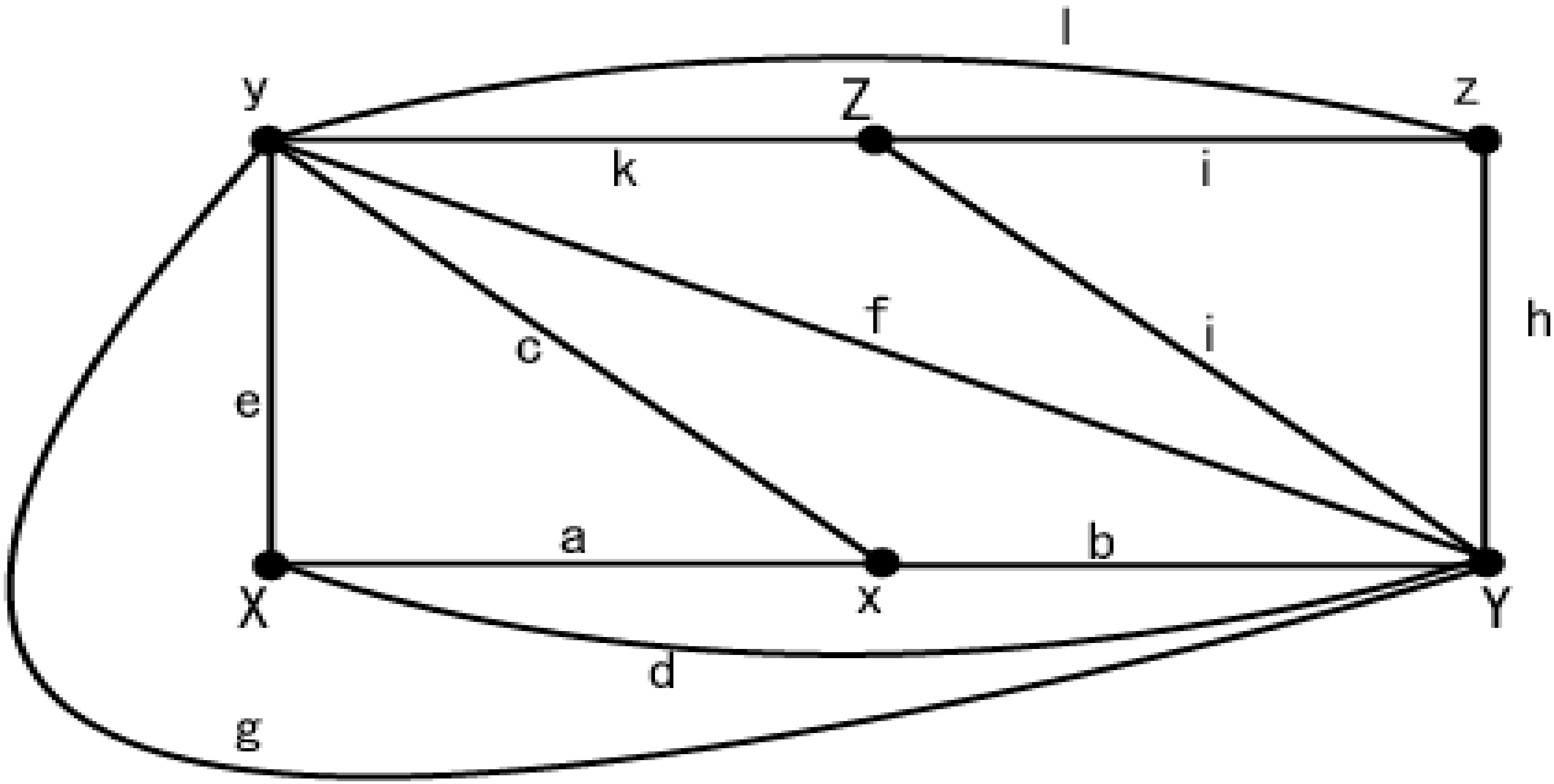

This case is special and an another approach is necessary.