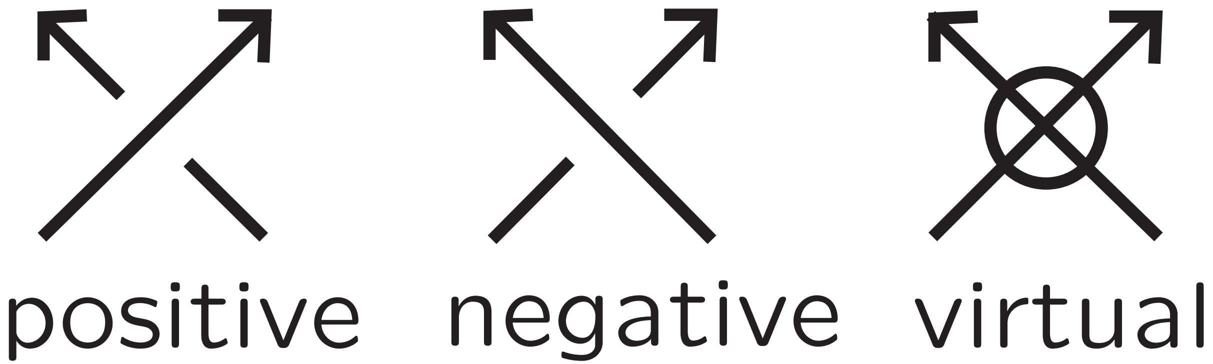
Constructing a table of virtual knots

Naoko Kamada

Osaka City University

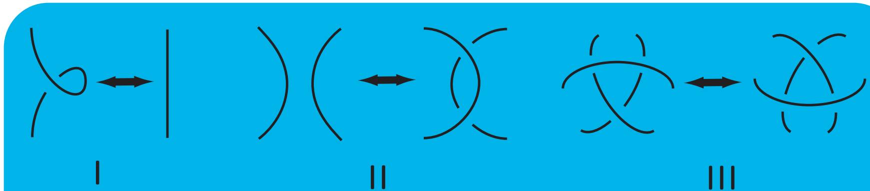
Virtual knot diagrams

A virtual knot/link diagram is a knot/link diagram which may have virtual crossings.

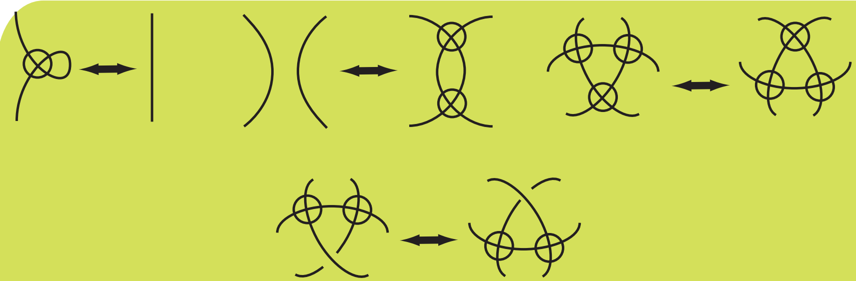


The equivalence class of a virtual knot

A virtual knot/link is the equivalence class of a virtual knot/link diagram under the generalized R-moves (generalized Reidemeister moves).



R-moves (Reidemeister moves)

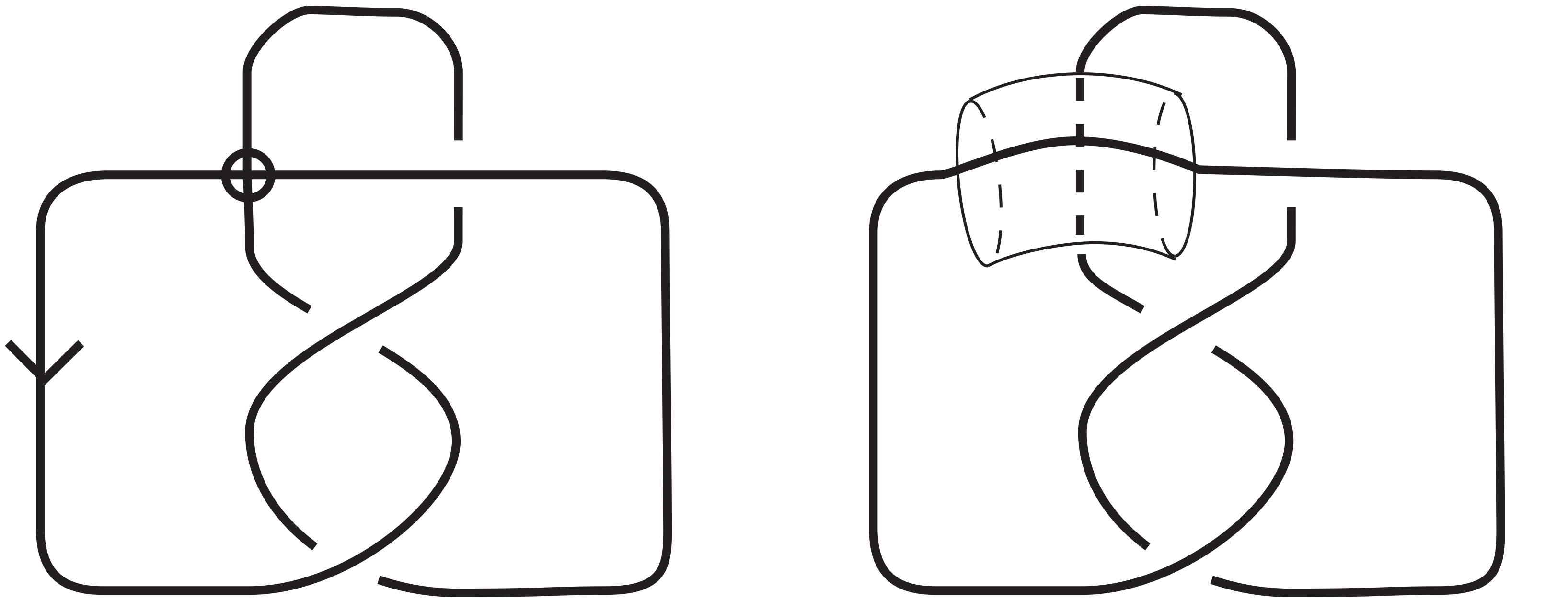


virtual R-moves
(virtual Reidemeister moves)

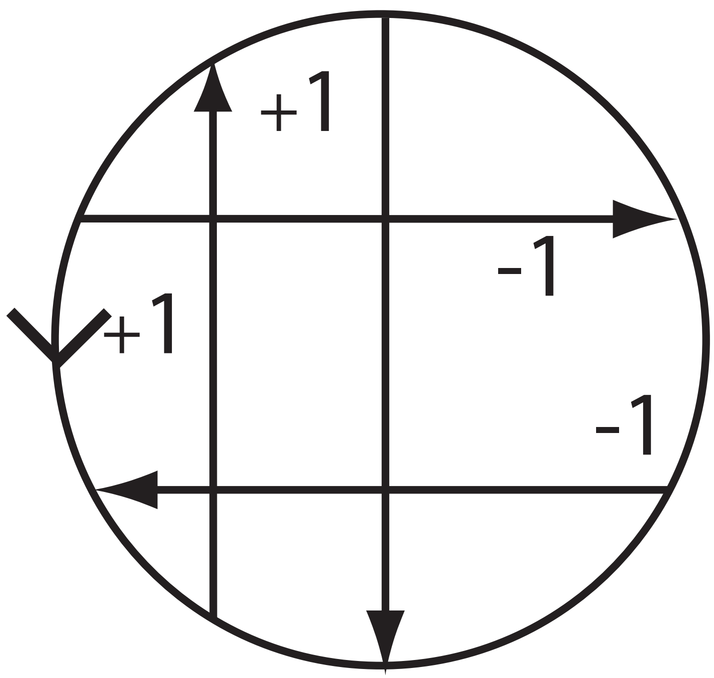
Fact (Kauffman)

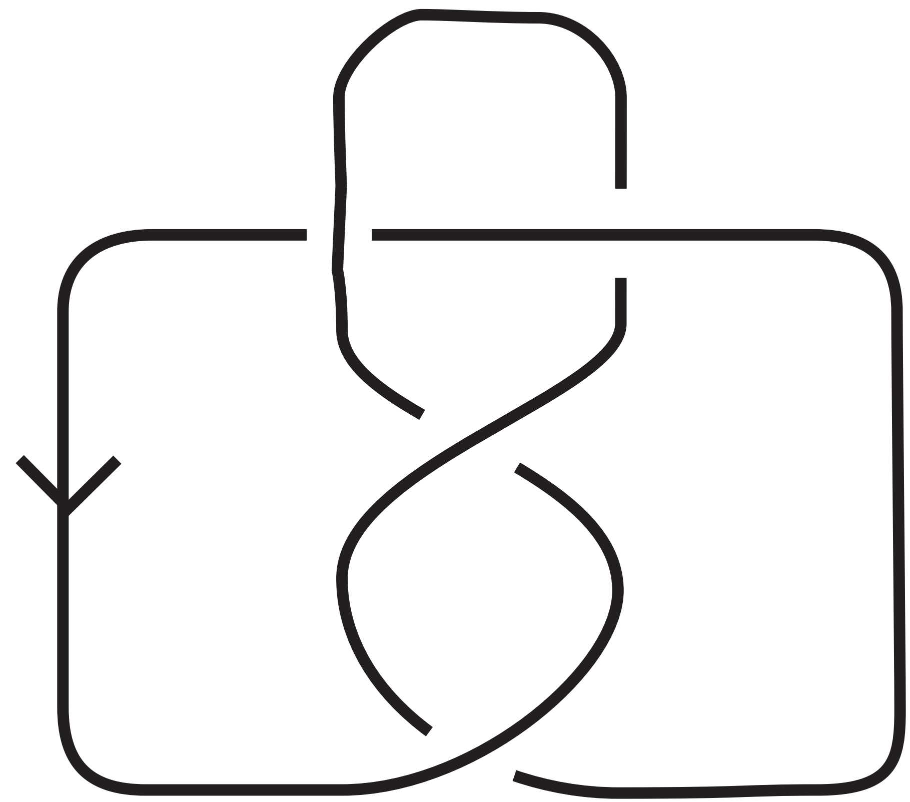
If two knot diagrams are equivalent under generalized R-moves, then they are equivalent under R-moves.

Link diagram on a surface

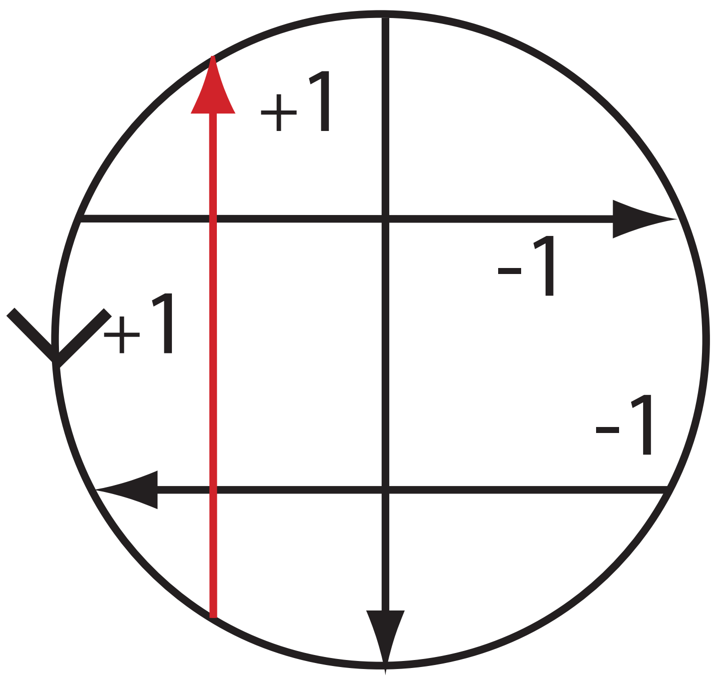


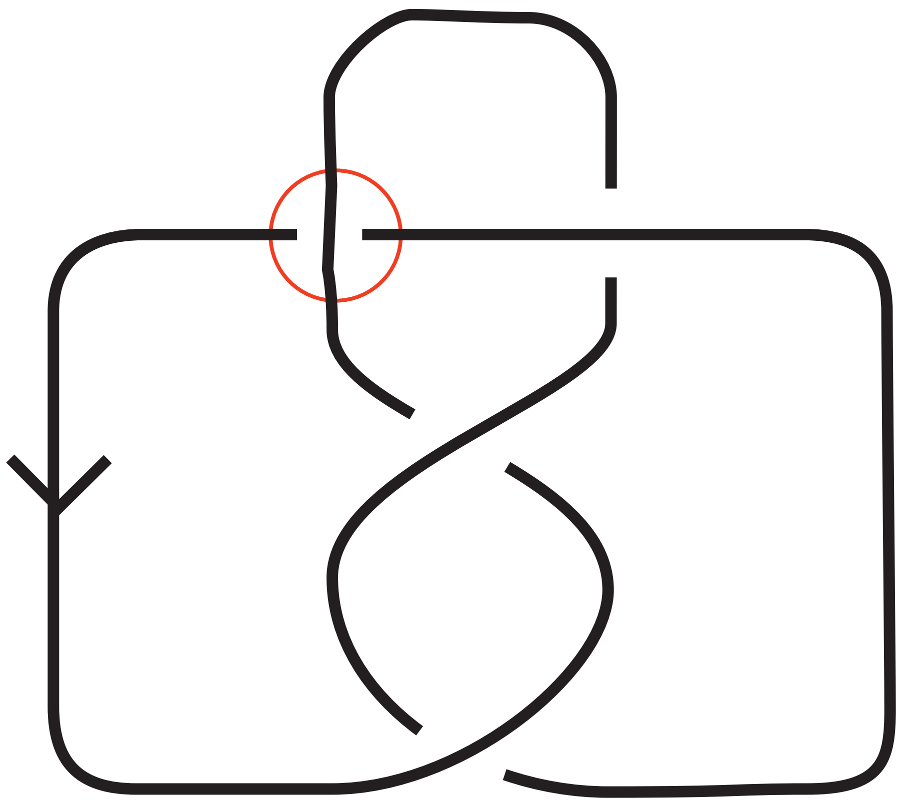
Gauss chord diagrams



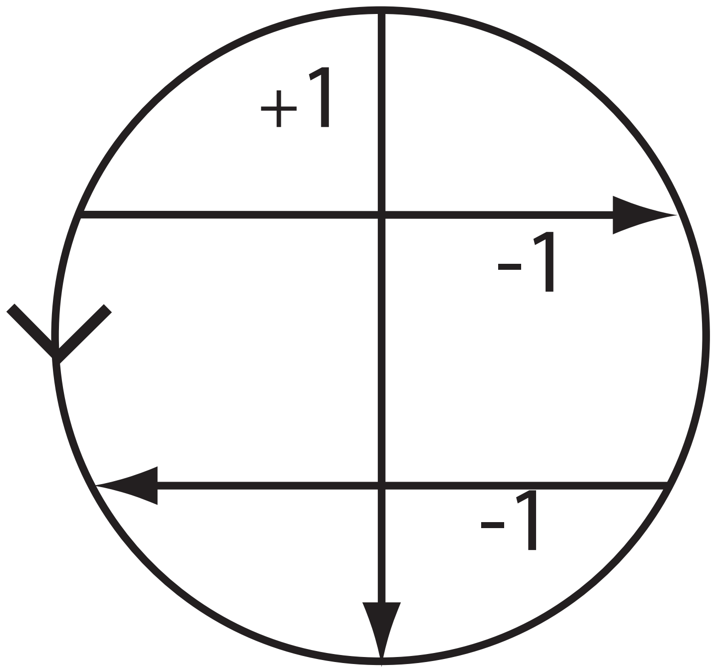


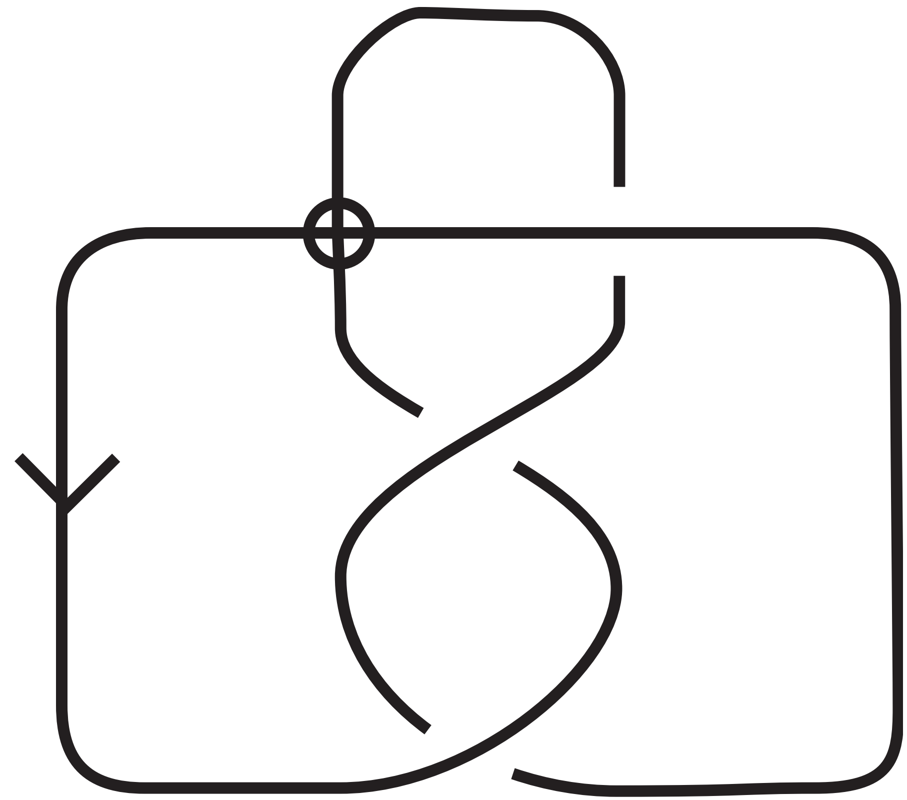
Gauss chord diagrams





Gauss chord diagrams





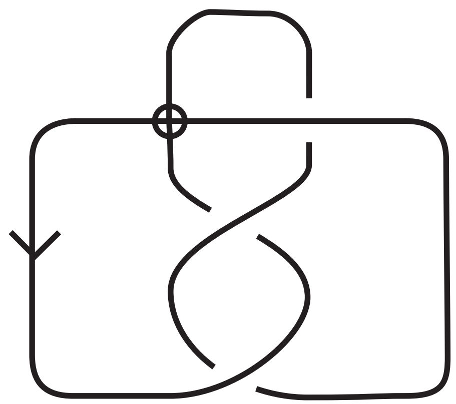
Fact (Kauffman)

$$\left\{\begin{array}{l}\text{virtual knot}\\ \text{diagrams}\end{array}\right\} \Big/ \begin{array}{l}\text{virtual}\\ \text{R-moves}\end{array} \Leftrightarrow \left\{\begin{array}{l}\text{Gauss chord}\\ \text{diagrams}\end{array}\right\}$$

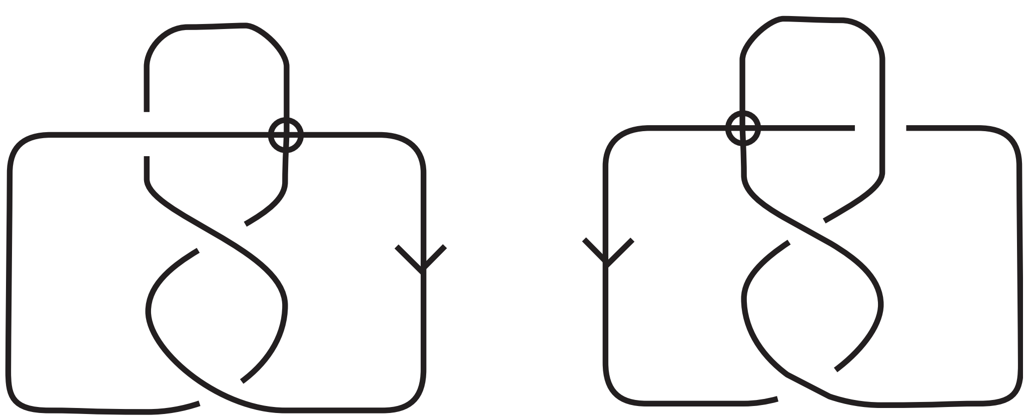
$$\begin{array}{l}\{\text{virtual knots}\}\\ \qquad\| \\ \left\{\begin{array}{l}\text{virtual knot}\\ \text{diagrams}\end{array}\right\} \Big/ \begin{array}{l}\text{generalized}\\ \text{R-moves}\end{array}\end{array} \Leftrightarrow \left\{\begin{array}{l}\text{Gauss chord}\\ \text{diagrams}\end{array}\right\} \Big/ \text{R-moves}$$

So, I constructed a table of Gauss chord diagrams which represent virtual knots with four real crossings (where a knot, mirror images and the reverse are regarded as equivalent) with some invariants of them.

Mirror images



D



the mirror images of *D*

Computer program

To list the data and caluculate invariants, I made a computer program.

Function of the program

· For a given number m, create prime Gauss chord diagrams whose number of chords is m.
· Get rid of Gauss chord diagrams which are transfered from another diagram by rotating, reversing orientations.
· Get rid of Gauss chord diagrams which are transfered from another diagram by a Reidemeister move III and which transfer to a diagram whose crossing number is less than m by a Reidemeister move II.
· Calculate the polynomial invariants of Gauss chord diagrams in the table.

Invariants

In the table of virtual knots (Gauss chord diagrams) with four real crossings (where a knot, mirror images and the reverse are regarded as equivalent), the following invariants are equipped.

Jones polynomials

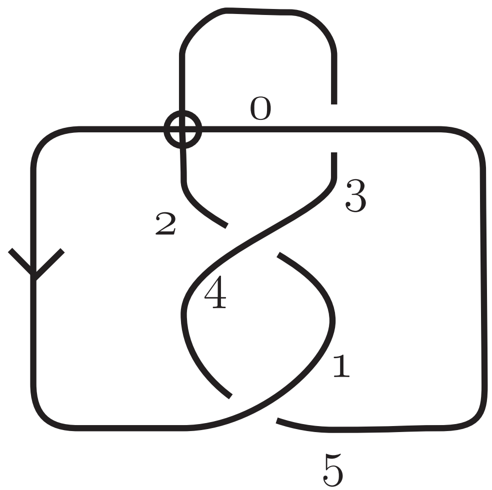
JKSS invariants

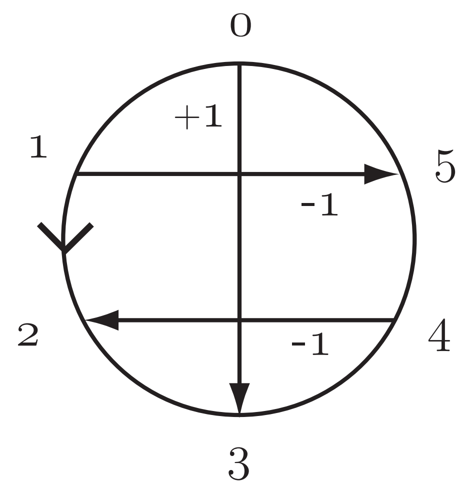
(Sawollek invariants, generalized Alexander polynomials

Miyazawa polynomials I

Miyazawa polynomials II

Gauss Chord diagram in the program





Gauss chord diagram $(0, 3), (1, 5), (4, 2)$

$$(1, -1, -1)$$

No	Chord	sign
1	(0, 2) (1, 3)	1, 1
2	(0, 3) (1, 4) (2, 5)	1, 1, 1
3	(0, 3) (4, 1) (2, 5)	1, 1, 1
4	(0, 3) (4, 1) (2, 5)	1, 1, -1
5	(0, 3) (1, 5) (2, 4)	1, 1, 1
6	(0, 3) (1, 5) (2, 4)	1, -1, -1
7	(0, 3) (1, 5) (4, 2)	1, 1, -1
8	(0, 3) (1, 5) (4, 2)	1, -1, -1
9	(0, 3) (1, 6) (2, 4) (5, 7)	1, 1, 1, 1
10	(0, 3) (1, 6) (2, 4) (5, 7)	1, 1, -1, 1
11	(0, 3) (1, 6) (2, 4) (5, 7)	1, 1, -1, -1
12	(0, 3) (1, 6) (2, 4) (5, 7)	1, -1, 1, 1
13	(0, 3) (1, 6) (2, 4) (5, 7)	1, -1, 1, -1
14	(0, 3) (1, 6) (2, 4) (5, 7)	1, -1, -1, -1
15	(0, 3) (1, 6) (2, 4) (7, 5)	1, 1, 1, -1

No	Chord	sign
16	(0, 3) (1, 6) (2, 4) (7, 5)	1, 1, -1, -1
17	(0, 3) (1, 6) (2, 4) (7, 5)	1, -1, 1, -1
18	(0, 3) (1, 6) (2, 4) (7, 5)	1, -1, -1, 1
19	(0, 3) (1, 6) (4, 2) (5, 7)	1, 1, 1, 1
20	(0, 3) (1, 6) (4, 2) (5, 7)	1, 1, 1, -1
21	(0, 3) (1, 6) (4, 2) (5, 7)	1, -1, 1, -1
22	(0, 3) (1, 6) (4, 2) (5, 7)	1, -1, -1, 1
23	(0, 3) (6, 1) (2, 4) (5, 7)	1, 1, 1, 1
24	(0, 3) (6, 1) (2, 4) (5, 7)	1, 1, 1, -1
25	(0, 3) (6, 1) (2, 4) (5, 7)	1, 1, -1, -1
26	(0, 3) (6, 1) (2, 4) (5, 7)	1, -1, 1, 1
27	(0, 3) (6, 1) (2, 4) (5, 7)	1, -1, 1, -1
28	(0, 3) (6, 1) (2, 4) (5, 7)	1, -1, -1, 1
29	(0, 3) (6, 1) (2, 4) (7, 5)	1, 1, 1, -1
30	(0, 3) (6, 1) (2, 4) (7, 5)	1, 1, -1, 1

No	Chord	sign
31	(0, 3) (6, 1) (2, 4) (7, 5)	1, -1, 1, -1
32	(0, 3) (6, 1) (2, 4) (7, 5)	1, -1, -1, 1
33	(0, 3) (6, 1) (2, 4) (7, 5)	1, -1, -1, -1
34	(0, 3) (6, 1) (4, 2) (7, 5)	1, 1, 1, 1
35	(0, 3) (6, 1) (4, 2) (7, 5)	1, 1, -1, -1
36	(0, 3) (6, 1) (4, 2) (7, 5)	1, -1, 1, 1
37	(0, 3) (6, 1) (4, 2) (7, 5)	1, -1, 1, -1
38	(0, 3) (1, 6) (2, 5) (4, 7)	1, 1, 1, -1
39	(0, 3) (1, 6) (2, 5) (4, 7)	1, -1, -1, 1
40	(0, 3) (1, 6) (2, 5) (7, 4)	1, 1, 1, 1
41	(0, 3) (1, 6) (2, 5) (7, 4)	1, -1, -1, 1
42	(0, 3) (1, 6) (5, 2) (4, 7)	1, 1, -1, -1
43	(0, 3) (1, 6) (5, 2) (4, 7)	1, -1, -1, 1
44	(0, 3) (6, 1) (2, 5) (4, 7)	1, 1, 1, 1
45	(0, 3) (6, 1) (2, 5) (4, 7)	1, 1, 1, -1

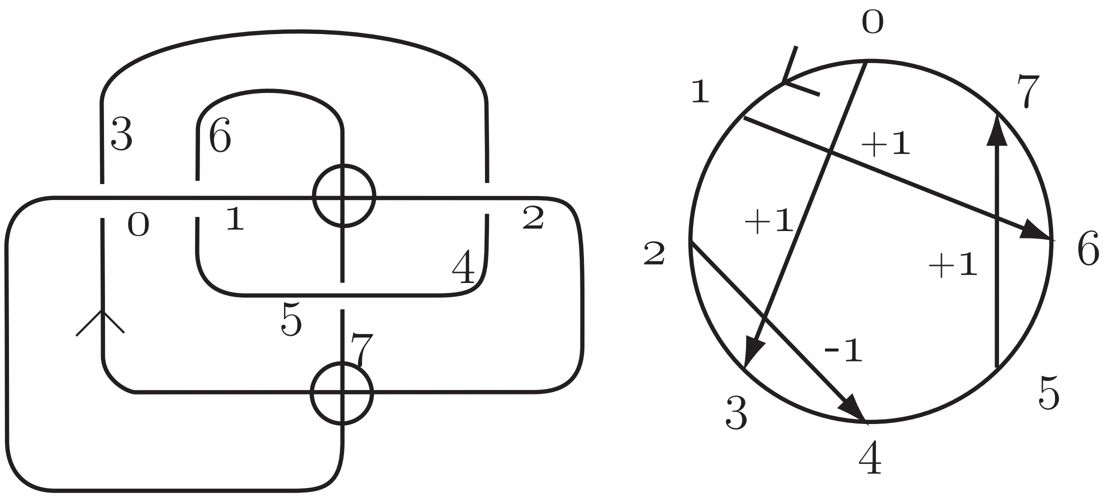
No	Chord	sign
46	(0, 3) (6, 1) (2, 5) (4, 7)	1, 1, -1, -1
47	(0, 3) (6, 1) (2, 5) (4, 7)	1, -1, -1, 1
48	(0, 4) (1, 5) (2, 6) (3, 7)	1, 1, 1, 1
49	(0, 4) (1, 5) (6, 2) (3, 7)	1, 1, 1, 1
50	(0, 4) (1, 5) (6, 2) (3, 7)	1, 1, 1, -1
51	(0, 4) (1, 5) (6, 2) (3, 7)	1, 1, -1, -1
52	(0, 4) (1, 5) (2, 7) (3, 6)	1, 1, 1, 1
53	(0, 4) (1, 5) (2, 7) (3, 6)	1, 1, -1, -1
54	(0, 4) (1, 5) (2, 7) (6, 3)	1, 1, 1, 1
55	(0, 4) (1, 5) (2, 7) (6, 3)	1, 1, 1, -1
56	(0, 4) (1, 5) (2, 7) (6, 3)	1, 1, -1, -1
57	(0, 4) (5, 1) (2, 7) (3, 6)	1, 1, 1, 1
58	(0, 4) (5, 1) (2, 7) (3, 6)	1, 1, -1, -1
59	(0, 4) (5, 1) (2, 7) (3, 6)	1, -1, 1, 1
60	(0, 4) (5, 1) (2, 7) (6, 3)	1, 1, -1, 1

No	Chord	sign
61	(0, 4) (5, 1) (2, 7) (6, 3)	1, -1, 1, 1
62	(0, 4) (5, 1) (2, 7) (6, 3)	1, -1, 1, -1
63	(0, 4) (5, 1) (7, 2) (6, 3)	1, 1, 1, 1
64	(0, 4) (5, 1) (7, 2) (6, 3)	1, -1, 1, 1
65	(0, 4) (1, 6) (2, 7) (3, 5)	1, 1, 1, 1
66	(0, 4) (1, 6) (2, 7) (3, 5)	1, 1, 1, -1
67	(0, 4) (1, 6) (2, 7) (3, 5)	1, -1, -1, 1
68	(0, 4) (1, 6) (2, 7) (3, 5)	1, -1, -1, -1
69	(0, 4) (1, 6) (2, 7) (5, 3)	1, 1, 1, 1
70	(0, 4) (1, 6) (2, 7) (5, 3)	1, 1, 1, -1
71	(0, 4) (1, 6) (2, 7) (5, 3)	1, -1, -1, 1
72	(0, 4) (1, 6) (2, 7) (5, 3)	1, -1, -1, -1
73	(0, 4) (1, 6) (7, 2) (3, 5)	1, 1, 1, 1
74	(0, 4) (1, 6) (7, 2) (3, 5)	1, 1, 1, -1
75	(0, 4) (1, 6) (7, 2) (3, 5)	1, 1, -1, 1

No	Chord	sign
76	(0, 4) (1, 6) (7, 2) (3, 5)	1, 1, -1, -1
77	(0, 4) (1, 6) (7, 2) (3, 5)	1, -1, 1, 1
78	(0, 4) (1, 6) (7, 2) (3, 5)	1, -1, 1, -1
79	(0, 4) (1, 6) (7, 2) (3, 5)	1, -1, -1, 1
80	(0, 4) (1, 6) (7, 2) (3, 5)	1, -1, -1, -1
81	(0, 4) (6, 1) (2, 7) (3, 5)	1, 1, 1, 1
82	(0, 4) (6, 1) (2, 7) (3, 5)	1, 1, 1, -1
83	(0, 4) (6, 1) (2, 7) (3, 5)	1, 1, -1, 1
84	(0, 4) (6, 1) (2, 7) (3, 5)	1, 1, -1, -1
85	(0, 4) (6, 1) (2, 7) (3, 5)	1, -1, 1, 1
86	(0, 4) (6, 1) (2, 7) (3, 5)	1, -1, 1, -1
87	(0, 4) (6, 1) (2, 7) (3, 5)	1, -1, -1, 1
88	(0, 4) (6, 1) (2, 7) (3, 5)	1, -1, -1, -1
89	(0, 4) (2, 6) (1, 3) (5, 7)	1, 1, 1, 1
90	(0, 4) (2, 6) (1, 3) (5, 7)	1, 1, 1, -1

No	Chord	sign
91	(0, 4) (2, 6) (1, 3) (5, 7)	1, -1, 1, 1
92	(0, 4) (2, 6) (1, 3) (5, 7)	1, -1, 1, -1
93	(0, 4) (2, 6) (1, 3) (7, 5)	1, 1, 1, 1
94	(0, 4) (2, 6) (1, 3) (7, 5)	1, -1, 1, 1
95	(0, 4) (2, 6) (3, 1) (5, 7)	1, 1, 1, 1
96	(0, 4) (2, 6) (3, 1) (5, 7)	1, 1, 1, -1
97	(0, 4) (2, 6) (3, 1) (5, 7)	1, 1, -1, -1
98	(0, 4) (2, 6) (3, 1) (5, 7)	1, -1, 1, -1
99	(0, 4) (2, 6) (3, 1) (5, 7)	1, -1, -1, -1

An virtual knot from the table



$$(0, 3), (1, 6), (2, 4), (5, 7)$$
$$(1, 1, -1, 1)$$

Invariants

Jones polynomial :

$-A^{-2} + A^{-4} + 2A^{-6} + A^{-8} - A^{-10} - 1$

JKSS invariant: $(x-1)^2(y+1)(x+y)$

Miyazawa polynomial I:

$$(-\frac{1}{4}A^{-4} + \frac{1}{2}A^{-8} - \frac{1}{4}A^{-12})(t^2 + t^{-2})$$
$$+ (-\frac{1}{2}A^{-2} + A^{-6} - \frac{1}{2}A^{-10})(t + t^{-1})$$
$$+ \frac{3}{2}A^{-4} - \frac{1}{2}A^{-12}$$

Miyazawa polynomial II:

$-A^{-2} - A^{-4} + 2A^{-6} + 2A^{-8} - A^{-10} - 1$

Virtual crossing number

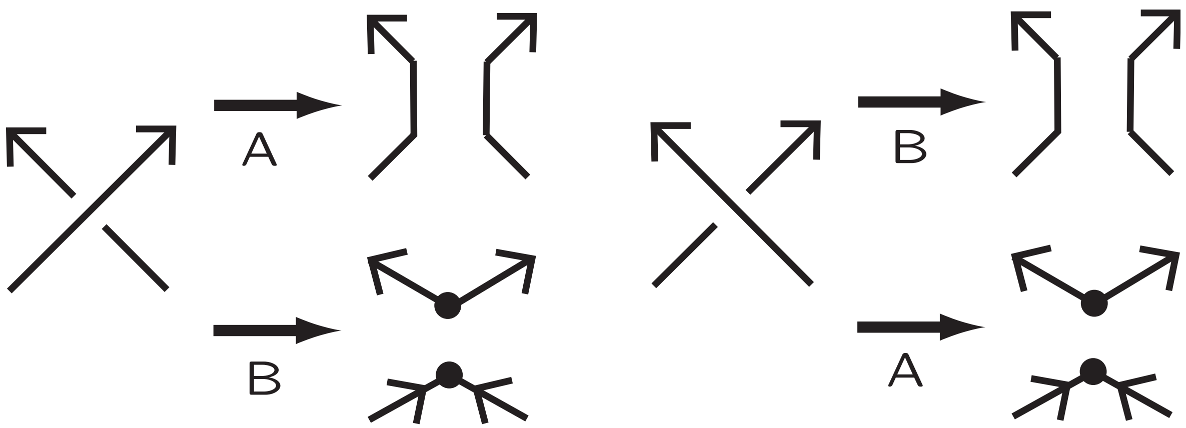
L: a virtual knot/link
$N_v(L)$:the virtual crossing number of L, which is the minimal number of virtual crossings among all diagrams representing L

Theorem 1 [Y. Miyazawa] $N_v(L)$ is equal to or greater than the maximal degree on t in Miyazawa polynomial I.

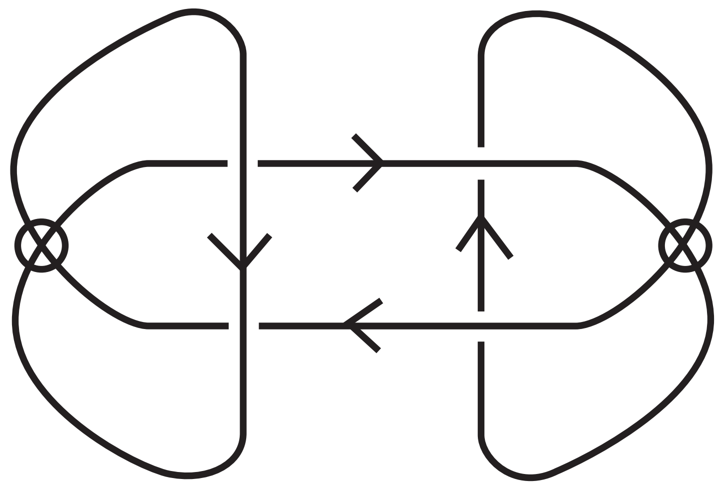
Splice

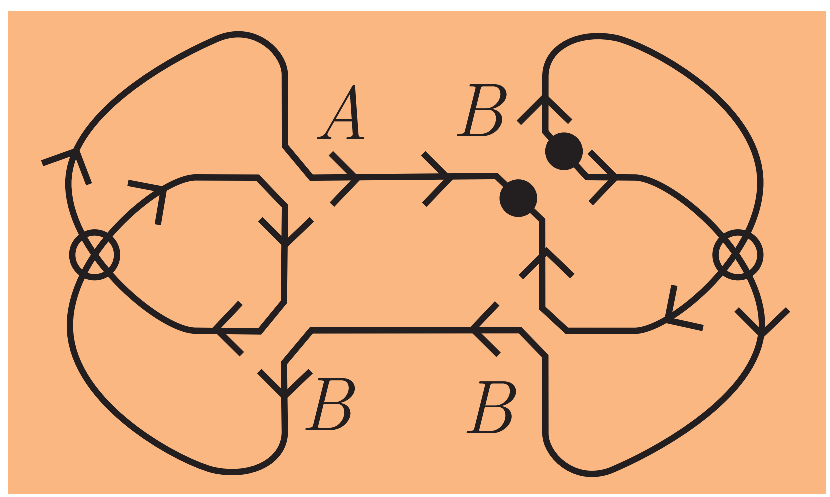
D : a virtual link diagram

A state of D means an assignment of A-splice or B-splice to each real crossing, or the diagram obtained from D by applying the splices. (A state is no longer a virtual link diagram.)



Example





Kauffman's f polynomial (Jones polynomial)

$$\begin{aligned} w(D) &= \text{writhe of } D \\ \natural(S) &= \sharp\{A\text{-splices in } S\} - \sharp\{B\text{-splices in } S\} \\ \mu(S) &= \sharp\{\text{loops of } S\} \end{aligned}$$

Kauffman's bracket polynomial

$$\langle D \rangle = \sum_S A^{\natural(S)}(-A^2 - A^{-2})^{\mu(S)-1}$$

Kauffman's f-polynomial

$$f(D) = (-A^3)^{-w(D)}\langle D \rangle$$

Miyazawa's polynomial

$g : \mathbb{Z} \to R$: a fixed map (R: a ring with 1)

Miyazawa's bracket (associated with g)

$$\langle D \rangle_g = \sum_S A^{\natural(S)} (-A^2 - A^{-2})^{\mu(S)-1} \ll S \gg_g$$

Miyazawa's polynomial (associated with g)

$$m_g(D) = (-A^3)^{-w(D)} \langle D \rangle_g$$

They are valued in $\mathbb{Q}[A, A^{-1}] \otimes R$.

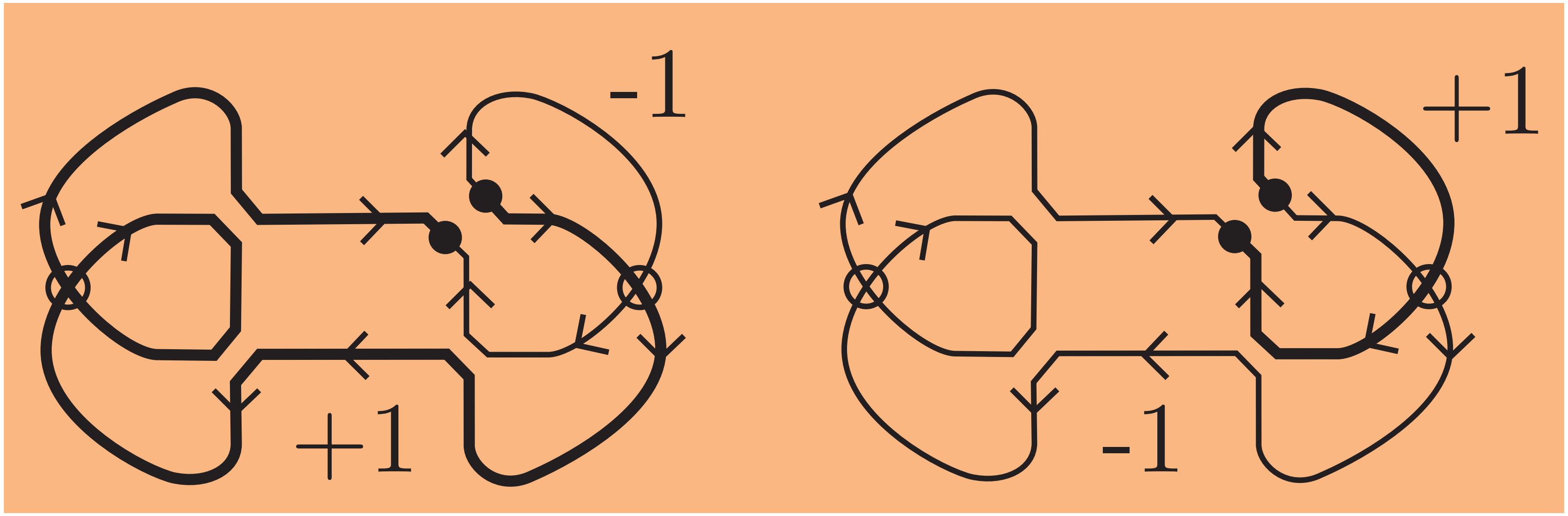
Remark

- If $g : Z \longrightarrow R$ is the unitary map $(g(n) = 1)$, then $m_g(D) = f(D)$.
- (Miyazawa polynomial I)
If $R = \mathbb{Z}[t, t^{-1}]$ and $g(n) = t^n$, then
$$m_g(D) \in \mathbb{Q}[A, A^{-1}, t, t^{-1}].$$
- (Miyazawa polynomial II)
If $R = \mathbb{Z}$ and $g(n) = |n|$, then
$$m_g(D) \in \mathbb{Q}[A, A^{-1}].$$

How to define $\ll S \gg_g$ for a state S.

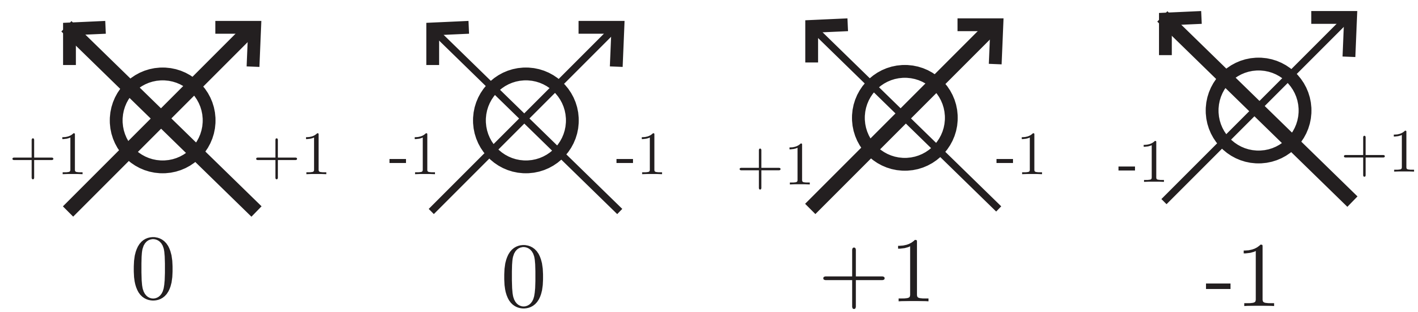
A weight map of a state S is a map τ:{edges of S}$\longrightarrow \{\pm 1\}$ such that $\tau(e) \neq \tau(e')$ for adjacent edges, e and e'.

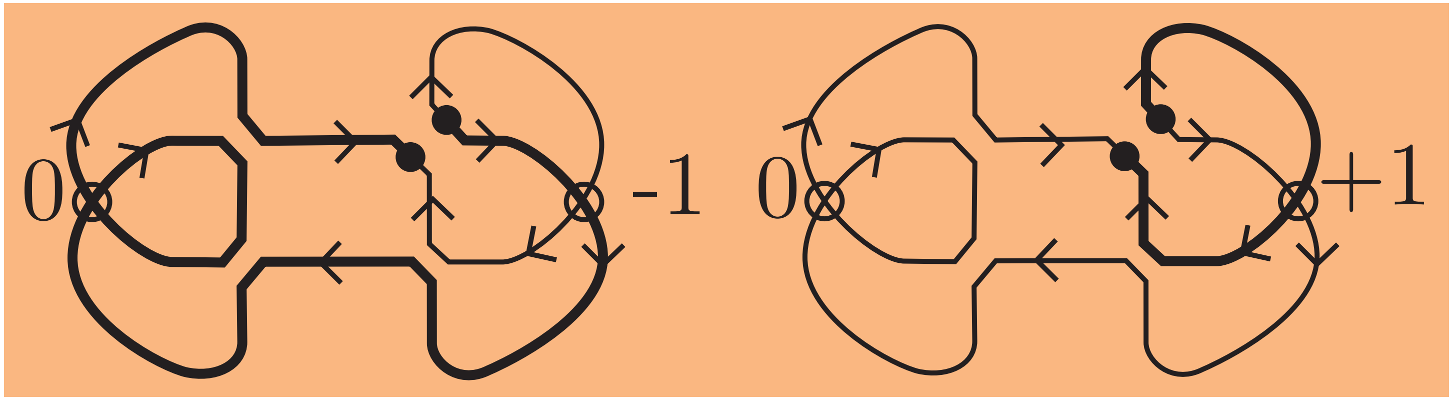
$\sharp$\{weight maps of S\} $= 2^{\mu(S)}$



virtual writhe

The virtual writhe $w^{\mathrm{virt}}(S, \tau)$ is the sum of signs of all virtual crossings.



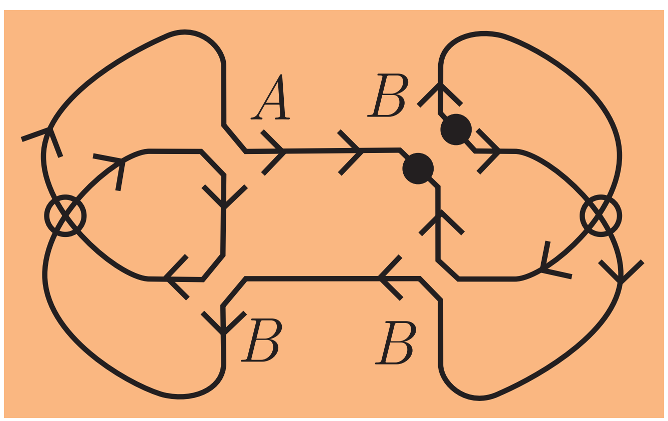


Double Bracket

$g : \mathbb{Z} \to R$: a fixed map

For a state S, we define

$$\ll S \gg_g = 2^{-\mu(S)} \sum_{\tau} g(w^{\mathrm{virt}}(S, \tau)).$$

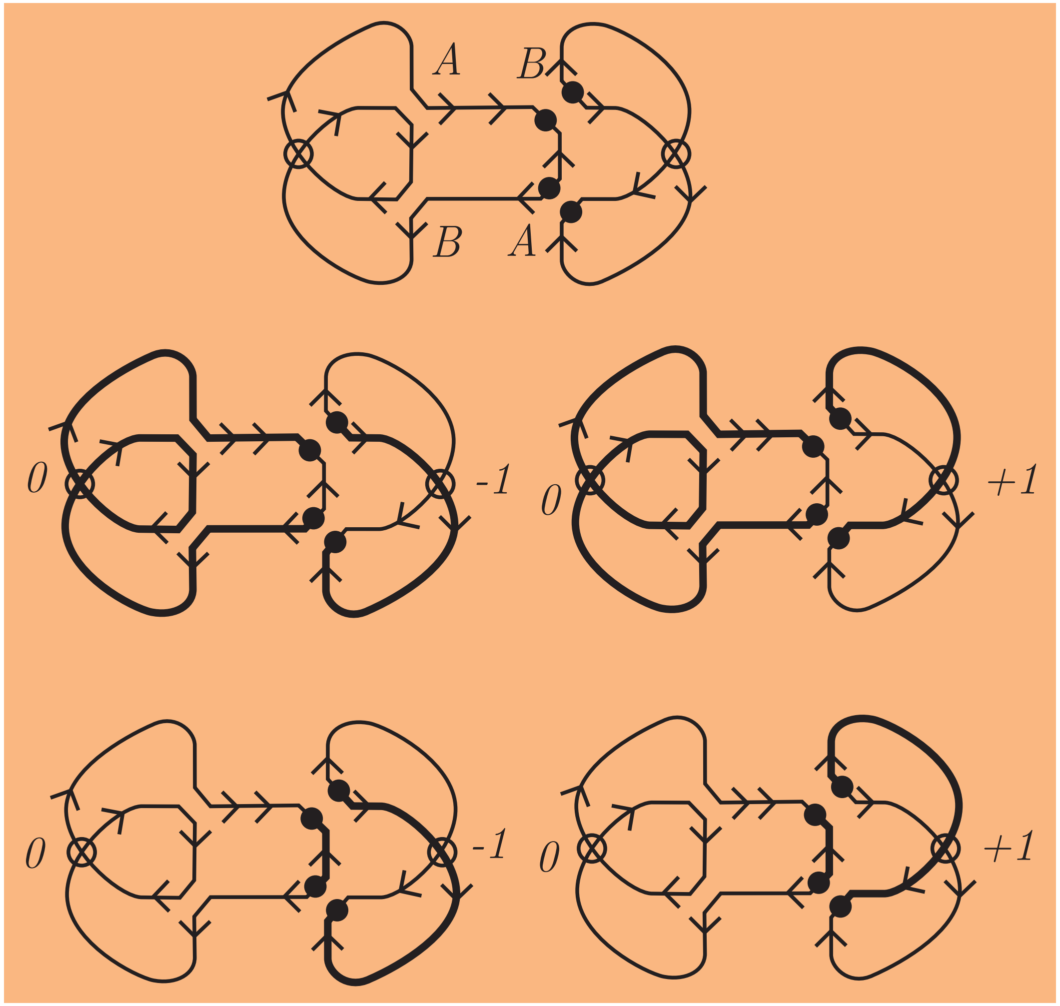


$$\ll S \gg_g = 2^{-1}(g(+1) + g(-1))$$

If $g(n) = t^n$, then

$$\ll S \gg_g = 2^{-1}(t + t^{-1})$$

An example of double bracket

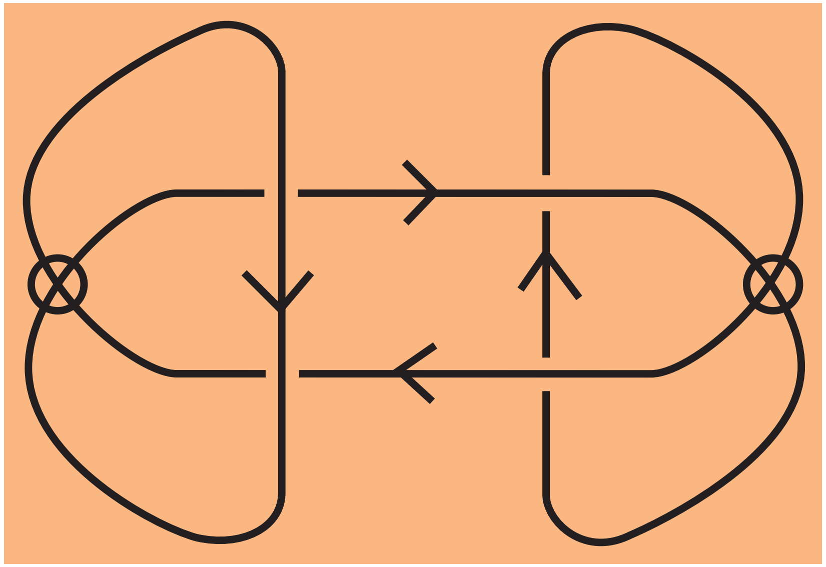


$\mu(S) = 2$
$\exists$ 4 weight maps $w^{\text{virt}}(S, f)$ are

$-1, +1, -1, +1.$
Thus, If $g(n) = t^n$, then

$$\ll S \gg_g = \frac{1}{4}(t^{-1} + t + t^{-1} + t) = \frac{1}{2}(t + t^{-1})$$

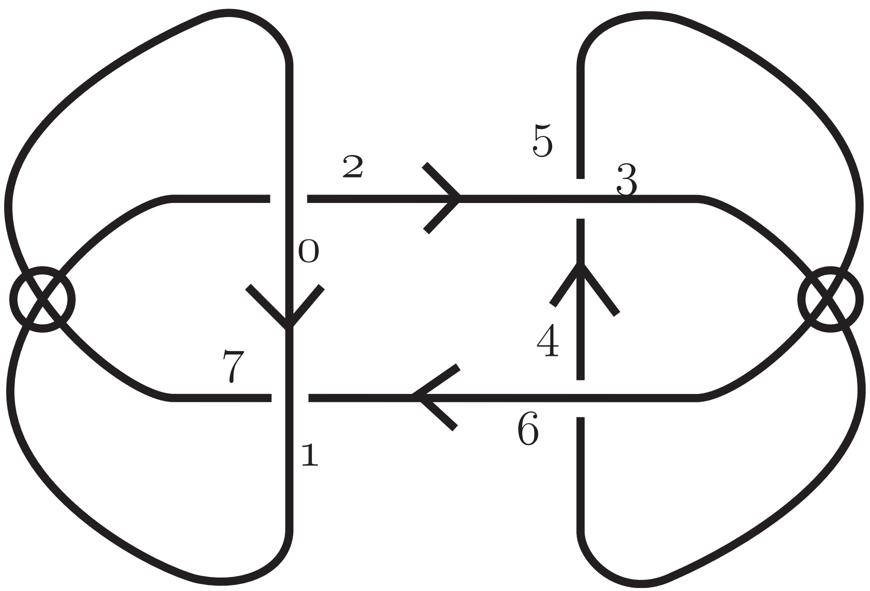
Kishino's knot



Miyazawa polynomial I of Kishino's knot:
$\frac{1}{2}(A^4 + A^{-4}) - \frac{1}{4}(A^2 - A^{-2})^2(t^2 + t^{-2})$
We see that the virtual crossing number of Kishino's knot is 2.

Compute Miyazawa polynomial

$$\left\{\text{virtual knot diagrams}\right\} \Big/ \text{virtual R-moves} \Leftrightarrow \left\{\text{Gauss chord diagrams}\right\}$$



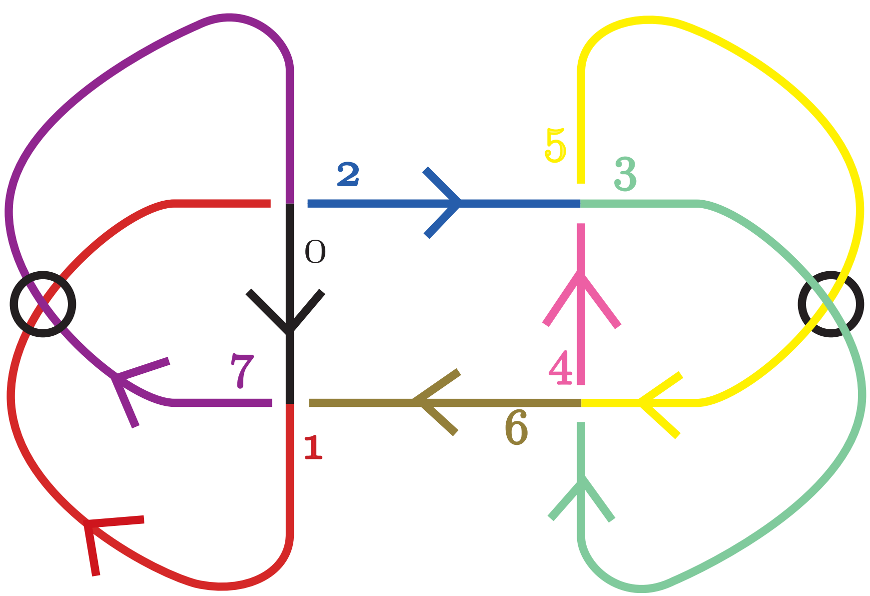
Gauss chord diagram

$(0,2)(1,7)(3,5)(6,4)$

$(1,-1,1,-1)$

Compute Miyazawa polynomial

$$\left\{\text{virtual knot diagrams}\right\} \Big/ \text{virtual R-moves} \Leftrightarrow \left\{\text{Gauss chord diagrams}\right\}$$



Gauss chord diagram

$(0,2)(1,7)(3,5)(6,4)$

$(1,-1,1,-1)$

Construct virtual crossings II

Kishino's knot

$(0,2)(1,7)(3,5)(6,4)$

$(1,-1,1,-1)$

Virtual crossings

$(0,1)_v, (0,7)_v, (1,2)_v,$
$(1,7)_v, (2,4)_v, (2,6)_v,$
$(2,7)_v, (3,4)_v, (3,5)_v,$
$(3,6)_v, (4,5)_v, (4,6)_v,$
$(5,6)_v$