Computations of 3-parallel HOMFLY polynomials of 6-braids.

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November 30, 2005

Jones Polynomial

V. Jones discovered a two-variable invariant $X_L(q,\lambda)$ of an oriented link L given by the following formula

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e tr(\pi(\alpha)),$$

where α is element of the braid group B_n with $\hat{\alpha} = L$, e being the exponent sum of α as a word on the σ_i's and π is the representation of B_n in the Hecke algebra $H(q,n)$, $\sigma_i \to g_i$.

Y is a Young diagram and tr_Y be the trace on the Hecke algebra (usual trace) on the image in the representation π_Y.

$$tr(x) = \sum_{Y} W_Y(q,\lambda) tr_Y(x)$$

where

$$W_Y(q,\lambda) = S(q,\lambda)/Q(q)$$

$Q(q)$ is a product of $(1-q^h)$ where h is a hook length.

$S(q,\lambda)$ is defined as a product of $(q^j - \lambda q^i)$, where i and j are the column and row number.

Let $\boldsymbol{Y}$ be a Young diagram $(\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_k})$. Then each node of $\boldsymbol{Y}$ has a value, called hook length, which is the total number of nodes which exist on the right and downward direction. For an example, a Young diagram (10, 4, 1) has the following hook length.

12	10	9	8	6	5	4	3	2	1
5	3	2	1						
1									

$1-\lambda q$	$q-\lambda q$	$q^2-\lambda q$	$q^3-\lambda q$	$\cdots$
$1-\lambda q^2$	$q-\lambda q^2$	$q^2-\lambda q^2$	$\cdots$	
$1-\lambda q^3$	$q-\lambda q^3$	$\cdots$	$\cdots$	
$1-\lambda q^4$	$\vdots$	$\vdots$		
$\vdots$				

For the Young diagram showed above, functions $Q(q)$ and $S(q,\lambda)$ are defined as follows:

$$
\begin{aligned}
Q(q) &= (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5) \\
&\times (1-q^6)(1-q^8)(1-q^9)(1-q^{10})(1-q^{12}) \\
&\times (1-q)(1-q^2)(1-q^3)(1-q^5)(1-q) \\
S(q,\lambda) &= (1-\lambda q)(q-\lambda q)(q^2-\lambda q)(q^3-\lambda q) \\
&\times (q^4-\lambda q)(q^5-\lambda q)(q^6-\lambda q)(q^7-\lambda q) \\
&\times (q^8-\lambda q)(q^9-\lambda q)(1-\lambda q^2)(q-\lambda q^2) \\
&\times (q^2-\lambda q^2)(q^3-\lambda q^2)(1-\lambda q^3).
\end{aligned}
$$

1. Polynomial invariants of Conway type can not distinuish two different mutant knots.

2. 3-parallel version of 2-variable Jones polynomials distinguish certain mutant knots[J. Murakami].

3. 3-parallel version of 2-variable Jones polynomial can distinguish Kinoshita-Terasaka knot (KT) and Conway knot (CK).

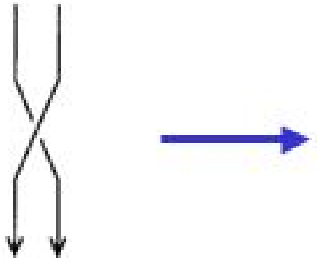
3-parallel version of 2-variable Jones polynomial

Let L be a link, α be an element of the braid group B_n with $\hat{\alpha} = L$, and β be the 3-parallel version of α. Then the following Laurent polynomial

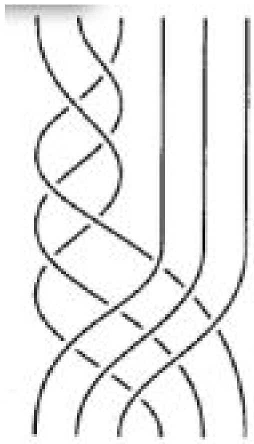
$$X_L^{(3)}(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e tr(\pi(\beta))$$

is a polynomial invariant of L.

3-parallel version



σ_i



σ_i

Hecke algebra

Let $H(q,n)$ of type A_{n-1} be a C-algebra with a unit defined by the following relations:

$$\begin{aligned} g_i^2 &= (q-1)g_i + q, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\ g_i g_j &= g_j g_i, \quad |i-j| \geq 2. \end{aligned}$$

Each g_i is a standard generator of $H(q,n)$.

For the standard generators $\boldsymbol{\sigma_1, \sigma_2, ..., \sigma_{n-1}}$ of $\boldsymbol{B_n}$, we define the algebra homomorphism $\boldsymbol{\Psi_n^{(3)}}$: $\boldsymbol{CB_n \to H(q, 3n)}$ as follows:

$$\begin{aligned} \boldsymbol{\Psi_n^{(3)}(\sigma_i)} &= \boldsymbol{g(3i-2, 3i-1)^{-3}} \\ &\times \boldsymbol{g(3i, 3i+2) g(3i-1, 3i+1) g(3i-2, 3i)} \end{aligned}$$

where $\boldsymbol{g(i, j) = g_i g_{i+1} ... g_j}$ $\boldsymbol{(1 \leq i < j \leq n-1)}$.

Direct calculation of 3-parallel invariant is very difficult.

- Braid length is multiplyed by 12. For the braid of length r, we must calculate the products of $12 \times r$ matrices.

- Size of representation matrices become very large. (up to 16,336,320 for $H(q, 18)$).

1. Restrict the base of representation to the subspace U.

2. Set λ to be a some power of q, say $\lambda = q^3$.

Let $\boldsymbol{Y}$ be a Young diagram for the set $\boldsymbol{\Lambda(n)}$ of partitions of a positive integer $\boldsymbol{n}$, and let $\boldsymbol{X} = \{\boldsymbol{x_1, x_2, ..., x_s}\}$ be the collection of words induced from the standard Young tableaux generated by $\boldsymbol{Y}$.

For each element $\boldsymbol{x}$ of $\boldsymbol{X}$, define $\boldsymbol{I}$-invariant $\boldsymbol{I(x)}$ as the set of $\boldsymbol{i} \in \{\boldsymbol{1, 2, ..., n-1}\}$ such that the row containing $\boldsymbol{i}$ is above the one containing $\boldsymbol{i+1}$ in $\boldsymbol{x}$. Let Y be a Young diagram associated with $\boldsymbol{\Lambda(3n)}$, and $\boldsymbol{G(Y)}$ be the W-graph with the vertex set $\boldsymbol{V(Y)} = \{\boldsymbol{\chi_1, \chi_2, ..., \chi_s}\}$ labeled by $\boldsymbol{I(G(Y))} = \{\boldsymbol{I(\chi_1), I(\chi_2), ..., I(\chi_s)}\}$ corresponding to $\boldsymbol{Y}$, where $\boldsymbol{I(\chi_i)}$ is the $\boldsymbol{I}$-invariant of $\boldsymbol{\chi_i}$.

Define a subset $V^3(Y)$ of $V(Y)$ as follows:

Each vertex χ in $V(Y)$ is included in $V^3(Y)$ if and only if $I(\chi)$ contains all numbers k in $I = \{1, 2, ..., 3n\}$ with $k \equiv 2(mod\ 3)$ and no numbers k in I with $k \equiv 1(mod\ 3)$.

Let π_Y be the representation given by the W-graph $G(Y)$, let $\pi_Y^\sharp$ be the restriction to $\pi_Y \circ \Psi^{(3)}$ on the subspace U spanned by the basis corresponding to $V^3(Y)$. It is well known that $\pi_Y^\sharp$ gives a representation of B_n.

$X_L^{(3)}(q,\lambda)^*$ is an invariant of L.

$$X_L^{(3)}(q,\lambda)^* = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e$$

$$\times \sum_Y W_Y(q,\lambda)\omega_Y^{(3)}(\alpha),$$

where $W_Y(q,\lambda) = S(q,\lambda)/Q(q)$ and $\omega_Y^{(3)}$ is the trace of $\pi_Y^\sharp$.

4-parallel invariant

For the standard generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ of B_n, we define the algebra homomorphism $\Psi_n^{(4)}$: $CB_n \to H(q, 4n)$ as follows:

$$\begin{aligned} \Psi_n^{(4)}(\sigma_i) &= g(4i-3, 4i-1)^{-4} \\ &\times g(4i, 4i+3) g(4i-1, 4i+2) \\ &\times g(4i-2, 4i+1) g(4i-3, 4i) \end{aligned}$$

where $g(i,j) = g_i g_{i+1} ... g_j$ $(1 \leq i < j \leq n-1)$.

Computational results

Jones Polynomial of KT and CW

$$L_{KT} = \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_3^{-1} \sigma_2^2 \sigma_3^{-1} \sigma_1^{-4} \sigma_2^2$$

$$L_{CW} = \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_3^{-1} \sigma_2^2 \sigma_3^{-1} \sigma_1 \sigma_2^{-3}$$

$$V(L_{KT}) = V(L_{CW}) =$$

$$-\frac{1}{t^4} + \frac{2}{t^3} - \frac{2}{t^2} + \frac{2}{t} + t^2 - 2t^3 + 2t^4 - 2t^5 + t^6$$

3-parallel invariant of KT

$$\boldsymbol{X_{KT}^{(3)}(q,q^3)^* =}$$
$$\boldsymbol{\frac{1}{q^{34}}(-1+3q-5q^2+9q^3-12q^4+6q^5+9q^6}$$
$$\boldsymbol{-28q^7+58q^8-105q^9+144q^{10}}$$
$$\boldsymbol{-169q^{11}+191q^{12}-191q^{13}}$$
$$\boldsymbol{+156q^{14}-106q^{15}+57q^{16}}$$
$$\boldsymbol{+30q^{17}-141q^{18}+233q^{19}-325q^{20}+435q^{21}}$$
$$\boldsymbol{-530q^{22}+551q^{23}-530q^{24}+462q^{25}-309q^{26}}$$
$$\boldsymbol{+99q^{27}+109q^{28}-286q^{29}+447q^{30}-529q^{31}}$$
$$\boldsymbol{+517q^{32}-462q^{33}+395q^{34}-305q^{35}+206q^{36}}$$
$$\boldsymbol{-147q^{37}+92q^{38}-18q^{39}-47q^{40}+85q^{41}}$$
$$\boldsymbol{-131q^{42}+177q^{43}-181q^{44}+156q^{45}-129q^{46}}$$
$$\boldsymbol{+97q^{47}-55q^{48}+23q^{49}-4q^{50}-8q^{51}}$$
$$\boldsymbol{+12q^{52}-9q^{53}+5q^{54}-3q^{55}+q^{56})}$$

3-parallel invariant of CW

$$X_{CW}^{(3)}(q,q^3)^* =$$
$$\frac{1}{q^{34}}(-1+3q-5q^2+9q^3-13q^4+12q^5-7q^6$$
$$+16q^8-47q^9+76q^{10}-103q^{11}+138q^{12}$$
$$-161q^{13}+156q^{14}-136q^{15}+110q^{16}$$
$$-36q^{17}-71q^{18}+163q^{19}-251q^{20}+351q^{21}$$
$$-430q^{22}+429q^{23}-393q^{24}+330q^{25}-203q^{26}$$
$$+39q^{27}+109q^{28}-226q^{29}+341q^{30}-397q^{31}$$
$$+380q^{32}-340q^{33}+295q^{34}-221q^{35}+132q^{36}$$
$$-77q^{37}+22q^{38}+48q^{39}-100q^{40}+115q^{41}$$
$$-131q^{42}+147q^{43}-128q^{44}+90q^{45}-61q^{46}$$
$$+39q^{47}-13q^{48}-5q^{49}+12q^{50}-14q^{51}$$
$$+13q^{52}-9q^{53}+5q^{54}-3q^{55}+q^{56})$$

Difference of invariants

$$X_{KT}^{(3)}(q,q^3)^* - X_{CW}^{(3)}(q,q^3)^* =$$

$$\frac{1}{q^{30}}(-1+q)^{11}(1+2q+2q^2+q^3)^3(1-q+2q^2-q^3+q^4)$$

$$\times(1+q^2+q^4+q^6+q^8+q^{10}+q^{12})^2$$

Jones polynomial of 6-braids

$$L_1 = aBcdeBcdbCbaaBBCDEcDc$$

$$L_2 = abCDEcDcBcdeBcdbCaBBa$$

where $a = \sigma_1, \ \ldots, \ A = \sigma_1^{-1}, \ \ldots$

$$V(L_1) = V(L_2) = -\frac{1}{t^9} + \frac{3}{t^8} - \frac{5}{t^7} + \frac{7}{t^6} - \frac{7}{t^5} + \frac{6}{t^4} - \frac{5}{t^3} + \frac{3}{t^2} - \frac{1}{t} + t$$

Jones polynomial of 6-braids

$$L_3 = AABcdeBCdbCAbAbCDECDc$$
$$L_4 = ABcdaBcBBaBaBCDEBCDBCbAbcde$$

$$V(L_3) = V(L_4) = -9 + \frac{3}{t} + 20t - 32t^2 + 42t^3 - 48t^4$$
$$+48t^5 - 42t^6 + 32t^7 - 20t^8$$
$$+10t^9 - 4t^{10} + t^{11}$$

3-parallel invariant of 6-braids

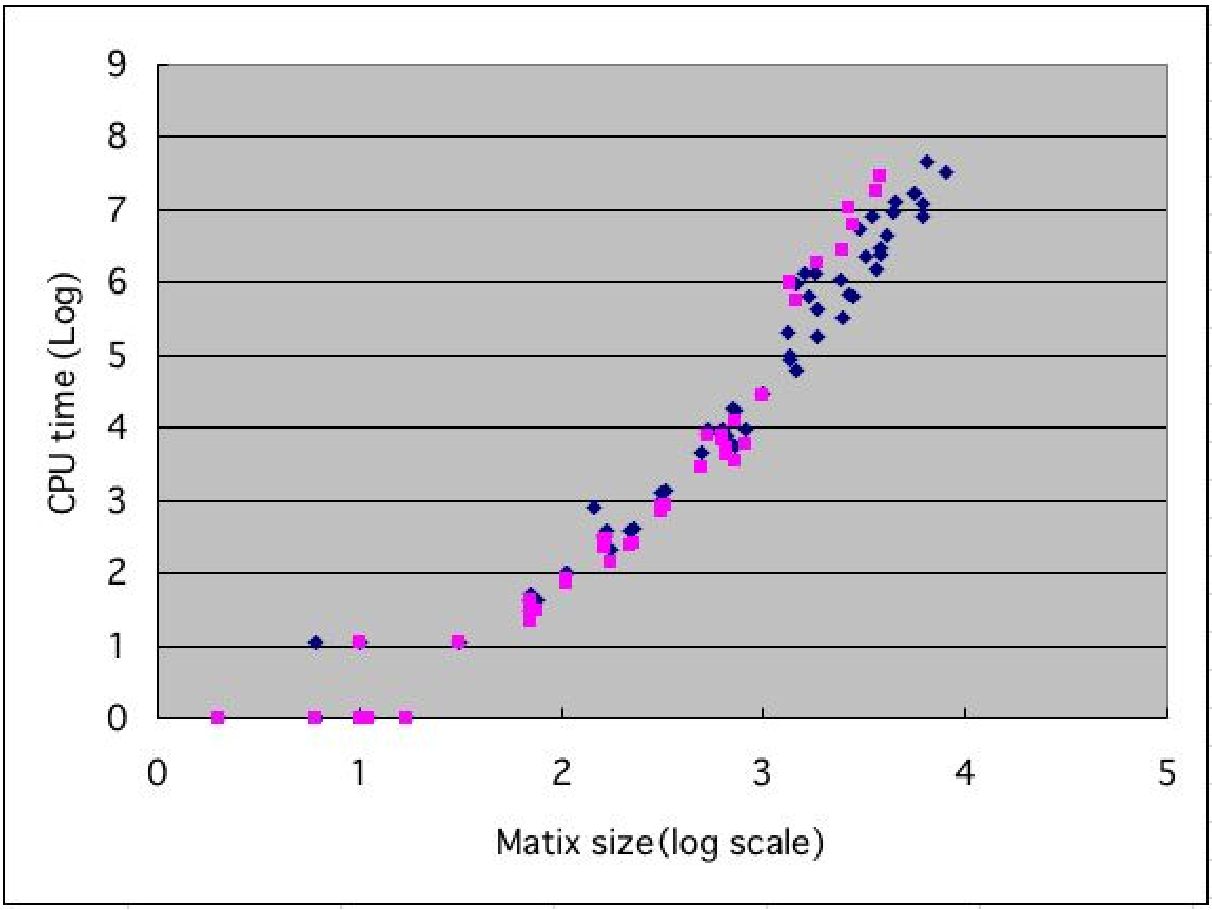
Failed to calculate 3-parallel invariant of 6-braids.

$$
\begin{aligned}
L_1 &= aBcdeBcdbCbaaBBCDEcDc \quad (ok) \\
L_2 &= abCDEcDcBcdeBcdbCaBBa \quad (failed) \\
L_3 &= AABcdeBCdbCAbAbCDECDc \quad (ok) \\
L_4 &= ABcdaBcBBaBaBCDEBCDBCbAbcde \\
&\qquad (failed)
\end{aligned}
$$

PowerMac G5 (6GB main memory)

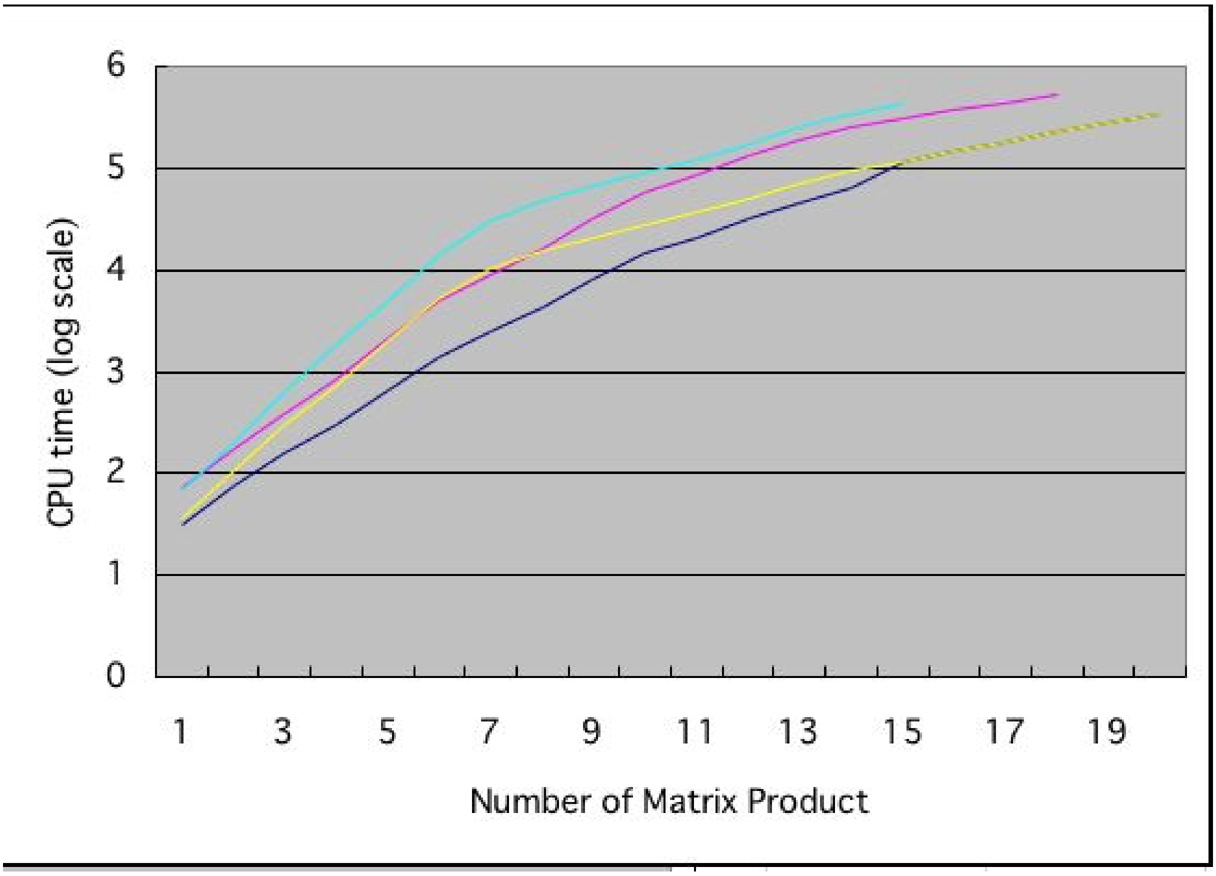
Cpu time to calculate the trace tr_Y for each Young diagram Y.

$$T \approx S^{2.5}$$



Cpu time to calculate the product of matrices.

$$T \approx 1.5^N$$



4-parallel invariant

```
ft4 = FourParallelPolynomialInvariant[1, KT]
```

4-parallel one variable invariant for [aCbCbbCA^4bb]

$$\frac{1}{q^{22}(1+q+q^2+q^3)}(1-3q+5q^2-6q^3+4q^4-q^5+2q^6-2q^7-3q^8+16q^9-30q^{10}+26q^{11}+q^{12}-41q^{13}+51q^{14}-15q^{15}-63q^{16}+143q^{17}-164q^{18}+103q^{19}+55q^{20}-234q^{21}+367q^{22}-364q^{23}+195q^{24}+72q^{25}-337q^{26}+450q^{27}-389q^{28}+172q^{29}+76q^{30}-242q^{31}+284q^{32}-195q^{33}+65q^{34}+62q^{35}-118q^{36}+105q^{37}-46q^{38}-16q^{39}+43q^{40}-38q^{41}+12q^{42}+4q^{43}-5q^{44}-q^{45}+3q^{46}-4q^{48}+5q^{49}-3q^{50}+q^{51})$$

```
fc4 = FourParallelPolynomialInvariant[1, CW]
```

4-parallel one variable invariant for [aCbCbbCaB^3]

$$\frac{1}{q^{22}(1+q+q^2+q^3)}(1-3q+5q^2-6q^3+4q^4-q^5+2q^6-2q^7-3q^8+16q^9-30q^{10}+26q^{11}+q^{12}-41q^{13}+51q^{14}-15q^{15}-63q^{16}+143q^{17}-164q^{18}+103q^{19}+55q^{20}-234q^{21}+367q^{22}-364q^{23}+195q^{24}+72q^{25}-337q^{26}+450q^{27}-389q^{28}+172q^{29}+76q^{30}-242q^{31}+284q^{32}-195q^{33}+65q^{34}+62q^{35}-118q^{36}+105q^{37}-46q^{38}-16q^{39}+43q^{40}-38q^{41}+12q^{42}+4q^{43}-5q^{44}-q^{45}+3q^{46}-4q^{48}+5q^{49}-3q^{50}+q^{51})$$

```
Simplify[ft4 - fc4]
```

0

4-parallel invariant

```
ft4 = FourParallelPolynomialInvariant[2, KT]
```

4-parallel one variable invariant for [aCbCbbCA^4bb]

$$\frac{1}{q^{41}(1+q)(1+q^2)}(1-2q-q^2+3q^3+q^4-5q^6-2q^7+6q^8+7q^9-15q^{11}-10q^{12}+18q^{13}+17q^{14}-6q^{15}-26q^{16}-17q^{17}+31q^{18}+25q^{19}+6q^{20}-52q^{21}-25q^{22}+62q^{23}+33q^{24}-4q^{25}-98q^{26}+16q^{27}+68q^{28}-3q^{29}-15q^{30}-72q^{31}-19q^{32}+108q^{33}+15q^{34}-122q^{35}+125q^{36}-138q^{37}+184q^{38}+23q^{39}-273q^{40}+374q^{41}-329q^{42}+197q^{43}+q^{44}-349q^{45}+478q^{46}-346q^{47}+102q^{48}+57q^{49}-232q^{50}+368q^{51}-185q^{52}-4q^{53}+60q^{54}-101q^{55}+183q^{56}-73q^{57}-74q^{58}+21q^{59}+8q^{60}+54q^{61}-27q^{62}-50q^{63}+42q^{65}+11q^{66}-17q^{67}-19q^{68}-4q^{69}+18q^{70}+12q^{71}-7q^{72}-14q^{73}+q^{74}+7q^{75}+5q^{76}-2q^{77}-5q^{78}+q^{80}+3q^{81}-q^{82}-2q^{83}+q^{84})$$

```
fc4 = FourParallelPolynomialInvariant[2, CW]
```

4-parallel one variable invariant for [aCbCbbCaB^3]

$$\frac{1}{q^{41}(1+q)(1+q^2)}(1-2q-q^2+3q^3+q^4-5q^6-2q^7+6q^8+7q^9-15q^{11}-10q^{12}+18q^{13}+17q^{14}-6q^{15}-26q^{16}-17q^{17}+31q^{18}+25q^{19}+6q^{20}-52q^{21}-25q^{22}+62q^{23}+33q^{24}-4q^{25}-98q^{26}+16q^{27}+68q^{28}-3q^{29}-15q^{30}-72q^{31}-19q^{32}+108q^{33}+15q^{34}-122q^{35}+125q^{36}-138q^{37}+184q^{38}+23q^{39}-273q^{40}+374q^{41}-329q^{42}+197q^{43}+q^{44}-349q^{45}+478q^{46}-346q^{47}+102q^{48}+57q^{49}-232q^{50}+368q^{51}-185q^{52}-4q^{53}+60q^{54}-101q^{55}+183q^{56}-73q^{57}-74q^{58}+21q^{59}+8q^{60}+54q^{61}-27q^{62}-50q^{63}+42q^{65}+11q^{66}-17q^{67}-19q^{68}-4q^{69}+18q^{70}+12q^{71}-7q^{72}-14q^{73}+q^{74}+7q^{75}+5q^{76}-2q^{77}-5q^{78}+q^{80}+3q^{81}-q^{82}-2q^{83}+q^{84})$$

```
ft4 - fc4
```

0

4-parallel invariant

```
Simplify[ff /. λ → q^2]
```

0

```
Simplify[ff /. λ → q^3]
```

0

```
Simplify[ff /. λ → q^4]
```

$$\Big((-1+q)^8 (1+q)^3 (-1+q^2)^3 (1+q^2)^2 (1+q+q^2) (1+q^2+q^4+q^6)^2 (1+2q^2+3q^3+q^4+3q^5+2q^6-q^7+q^8-q^9-2q^{10}+2q^{11}+2q^{12}+6q^{13}+9q^{14}+9q^{15}+11q^{16}+10q^{17}+7q^{18}+8q^{19}+4q^{20}+4q^{21}+5q^{22}+2q^{23}+4q^{24}+5q^{25}+2q^{26}+3q^{27}+2q^{28}+q^{30})\Big) \Big/ \left(q^{39} (1+2q+4q^2+5q^3+4q^4+2q^5+q^6)\right)$$

```
Simplify[ff /. λ → q^5]
```

$$-\frac{1}{q^{50}} \Big((-1+q)^{11} (1+q)^6 (1+q+2q^2+q^3+q^4)^2 (-1+2q-3q^2+q^3+q^4-5q^5+3q^6-2q^7-5q^8+3q^9-3q^{10}-5q^{11}+4q^{12}-5q^{13}-2q^{14}+3q^{15}-4q^{16}-4q^{17}+2q^{18}-6q^{19}-7q^{20}-7q^{21}-7q^{22}-17q^{23}-9q^{24}-22q^{25}-14q^{26}-21q^{27}-18q^{28}-26q^{29}-14q^{30}-21q^{31}-21q^{32}-16q^{33}-13q^{34}-18q^{35}-10q^{36}-9q^{37}-12q^{38}-5q^{39}-2q^{40}-10q^{41}+q^{42}+q^{43}-6q^{44}+3q^{45}+q^{46}-3q^{47}+4q^{48}-q^{49}-q^{50}+3q^{51}-2q^{52}+q^{53})\Big)$$

```
Simplify[ff /. λ → q^6]
```

$$-\Big((-1+q)^8 (-1+q^2)^3 (1+q^2)^2 (1+q+q^2) (1+2q+2q^2+q^3)^3 (-1-2q^2-5q^3+q^4-12q^5-8q^6-2q^7-31q^8-6q^9-14q^{10}-48q^{11}-19q^{13}-68q^{14}+34q^{15}-37q^{16}-57q^{17}+63q^{18}-36q^{19}-59q^{20}+111q^{21}-72q^{22}-54q^{23}+116q^{24}-132q^{25}-84q^{26}+72q^{27}-225q^{28}-158q^{29}-359q^{31}-215q^{32}-122q^{33}-433q^{34}- \ldots$$

RR1806601

$$
\begin{pmatrix}
-q^3 & & & & & & & & & \\
 & -q^3 & & & & & & & & \\
 & q\sqrt{q} & 1 & & & & & & & \\
 & & & -q^3 & & & & & & \\
 & & & q\sqrt{q} & 1 & & & & & \\
 & & & & & 1 & & & & \\
 & & & & & & -q^3 & & & \\
 & & & & & & q\sqrt{q} & 1 & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1
\end{pmatrix}
$$

RR1806602

$$\begin{pmatrix}
-q^3 & & & & & & & & & \\
q\sqrt{q} & 1 & q\sqrt{q} & & & & & & & \\
& & -q^3 & & & & & & & \\
& & & 1 & q\sqrt{q} & & & & & \\
& & & & -q^3 & & & & & \\
& & & & q\sqrt{q} & 1 & & & & \\
& & & & & & 1 & q\sqrt{q} & & \\
& & & & & & & -q^3 & & \\
& & & & & & & q\sqrt{q} & 1 & \\
& & & & & & & & & 1
\end{pmatrix}$$

RR1806603

$$
\begin{pmatrix}
1 & q\sqrt{q} & & & & & & & & \\
 & -q^3 & & & & & & & & \\
 & & -q^3 & & & & & & & \\
 & q\sqrt{q} & & 1 & & & & & & \\
 & & q\sqrt{q} & & 1 & q\sqrt{q} & & & & \\
 & & & & & -q^3 & & & & \\
 & & & & & & -q^3 & & & \\
 & & & & & & & 1 & q\sqrt{q} & \\
 & & & & & & & & -q^3 & \\
 & & & & & & & & q\sqrt{1} & 1
\end{pmatrix}
$$

RR1806604

$$
\begin{pmatrix}
1 & & & & & & & & & \\
& 1 & & q\sqrt{q} & & & & & & \\
& & 1 & & q\sqrt{q} & & & & & \\
& & & -q^3 & & & & & & \\
& & & & -q^3 & & & & & \\
& & q\sqrt{q} & & & 1 & & & & \\
& & & q\sqrt{q} & & & 1 & & & \\
& & & & q\sqrt{q} & & & 1 & & \\
& & & & & q\sqrt{q} & & & 1 & q\sqrt{q} \\
& & & & & & & & & -q^3
\end{pmatrix}
$$

$$\begin{pmatrix}
\mathbf{1} & & & & & & & & & \\
& \mathbf{1} & & & & & & & & \\
& & \mathbf{1} & & & & & & & \\
& & & \mathbf{1} & & & \boldsymbol{q\sqrt{q}} & & & \\
& & & & \mathbf{1} & & & \boldsymbol{q\sqrt{q}} & & \\
& & & & & \mathbf{1} & & & \boldsymbol{q\sqrt{q}} & \\
& & & & & & \boldsymbol{-q^3} & & & \\
& & & & & & & \boldsymbol{-q^3} & & \\
& & & & & & & & \boldsymbol{-q^3} & \\
& & & & & & & & & \boldsymbol{-q^3}
\end{pmatrix}$$