Coxeter groups, simplicial complexes and boundaries

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Definition. A *Coxeter group* is a group W having a presentation

$$\langle\, S \,|\, (st)^{m(s,t)} = 1 \text{ for } s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for all $s,t \in S$.

(2) $m(s,s) = 1$ for all $s \in S$, i.e., $s^2 = 1$.

(3) $m(s,t) \geq 2$ for all $s \neq t \in S$.

The pair (W,S) is called a *Coxeter system*.

If, in addition,

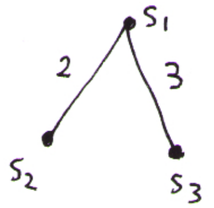
(4) $m(s,t) = 2$ or ∞ for all $s \neq t \in S$,

then (W,S) is said to be *right-angled*.

Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. It is known that the pair (W_T,T) is also a Coxeter system.

$$W = \left\langle \{s_1, s_2, s_3\} \;\middle|\; \begin{array}{l} s_1^2 = s_2^2 = s_3^2 = 1 \\ (s_1 s_2)^2 = 1 \\ (s_1 s_3)^3 = 1 \end{array} \right\rangle$$: Coxeter group

$(W, \{s_1, s_2, s_3\})$: Coxeter system



Coxeter diagram

$$\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & 1 & 2 & 3 \\ s_2 & 2 & 1 & \infty \\ s_3 & 3 & \infty & 1 \end{array}$$

Coxeter matrix

Definition. Let (W, S) be a Coxeter system. We define a simplicial complex $L(W, S)$ by the following conditions:

(1) the vertex set of $L(W, S)$ is S, and

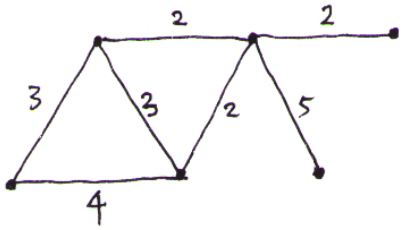
(2) for each nonempty subset T of S,

T spans a simplex of $L(W, S)$ if and only if W_T is finite.

For each nonempty subset T of S, $L(W_T, T)$ is a subcomplex of $L(W, S)$.

ex

(W, S)



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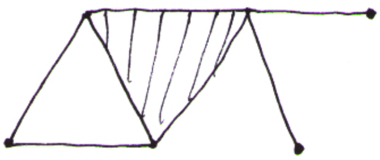
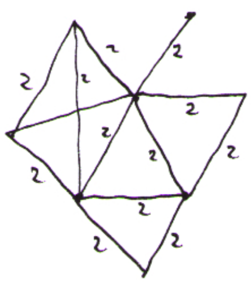
L(W, S)



(W, S)



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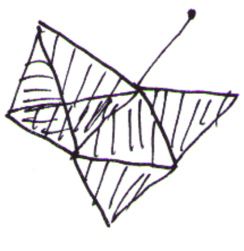
L(W, S)



Definition. A simplicial complex L is called a *flag complex*, if for each vertex subset $T \subset L^0$ such that $|s,t| \in L$ for any $s \neq t \in T$, T spans a simplex of L.

Remark.

(1) If (W,S) is a right-angled Coxeter system, then $L(W,S)$ is a flag complex.

(2) For a flag complex L, there exists a right-angled Coxeter system (W,S) such that $L(W,S) = L$.

(3) For any simplicial complex K, the barycentric subdivision sd K is a flag complex.

The following problem is open.

Problem. For any simplicial complex L, is there a Coxeter system (W,S) such that $L(W,S) = L$?

Definition. Let (W, S) be a Coxeter system.

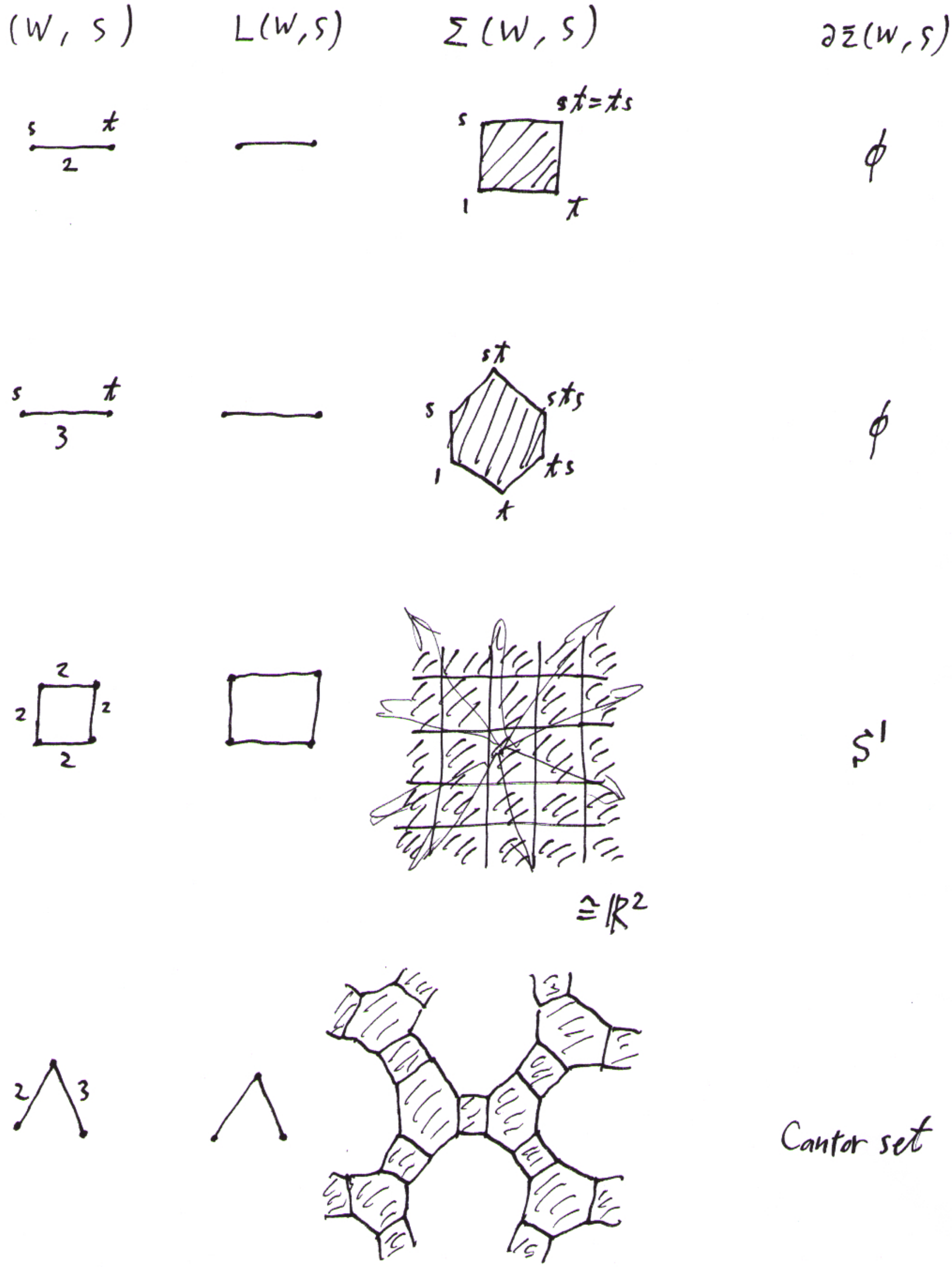
We define a cell complex $\Sigma(W, S)$ as follows:

(1) In the case W is finite, $\Sigma(W, S)$ is the convex hull of the Cayley graph of W with respect to S in $\mathbb{R}^{|S|}$.

(2) In general, the vertex set of $\Sigma(W, S)$ is W and

$$\Sigma(W, S) = \bigcup \{ w\Sigma(W_T, T) \mid w \in W,\ T \subset S,\ W_T \text{ is finite} \}.$$

We note that the 1-skeleton of $\Sigma(W, S)$ is the Cayley graph of (W, S).


(W, S)
L(W,S)
Σ(W, S)
∂Σ(W,S)
s
t
2
s
st=ts
1
t
φ
s
t
3
st
s
sts
1
ts
t
φ
2
2
2
2
S¹
≅ ℝ²
2
3
Cantor set

Moussong proved that $\Sigma(W,S)$ with the natural metric is a CAT(0) space. The CAT(0) space $\Sigma(W,S)$ can be compactified by adding its boundary $\partial\Sigma(W,S)$.

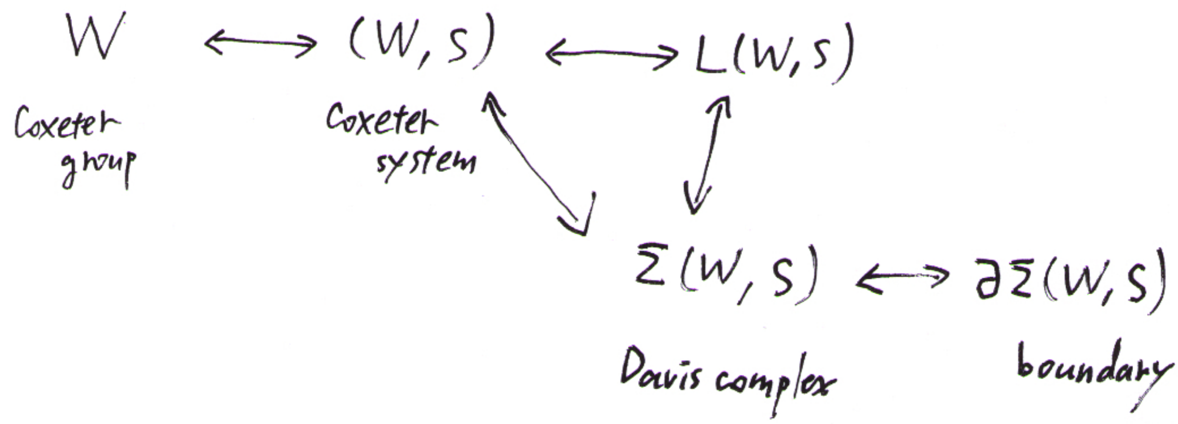
$$\partial\Sigma(W,S) = \{\xi : [0,\infty) \to \Sigma(W,S)\, \text{geodesic ray} \mid \xi(0) = x_0\}$$

The boundary $\partial\Sigma(W,S)$ is called the *boundary* of the Coxeter system (W,S). It is unknown whether the following conjecture holds.

$\partial\Sigma(W,S)$: compact metric space

Rigidity Conjecture (Dranishnikov).

Isomorphic Coxeter groups will have homeomorphic boundaries.


W
Coxeter group
(W, S)
Coxeter system
L(W, S)
Σ(W, S)
Davis complex
∂Σ(W, S)
boundary

Definition. Let X be a compact metric space and let G be an abelian group. The *cohomological dimension of* X *with respect to* G is defined as

$$\dim_G X = \sup\{i \mid \check{H}^i(X, A; G) \neq 0 \text{ for some closed subset } A \subset X\},$$

where $\check{H}^*$ denote the Čech cohomology.

Remark.

(1) If the dimension of X is finite, then $\dim X = \dim_{\mathbb{Z}} X$.

(2 If X is a polyhedron or manifold, then for any non-trivial abelian group G, $\dim X = \dim_G X$.

Theorem (Pontoryagin, Kuzminov, Dranishnikov)

For any $n \in \mathbb{N}$ there exists a compact metric space X_n such that $\dim X_n = n$ and $\dim_{\mathbb{Q}} X_n = 1$.

$n = 2$ Pontoryagin (1930's)

$n = 3$ Kuzminov (1960's)

$n \geq 4$ Dranishnikov (1980's)

Problem (Dranishnikov)

For each $n \in \mathbb{N}$, does there exist a Coxeter system (W, S) such that $\dim \partial\Sigma(W, S) = n$ and $\dim_{\mathbb{Q}} \partial\Sigma(W, S) = 1$?

n = 2 Dranishnikov (1997)

Fact:

$$H^*(W; RW) \cong H_c^*(\Sigma(W,S); R) \cong \check{H}^{*-1}(\partial\Sigma(W,S); R),$$

where H_c^* and $\check{H}^*$ denote the cohomology with compact supports and the reduced Čech cohomology, respectively.

Theorem (Bestvina and Mess 1991).

Let R a commutative ring with 1. Then

$$\dim_R \partial\Sigma(W,S) = \operatorname{vcd}_R W - 1,$$

where $\operatorname{vcd}_R W$ is the *virtual cohomological dimension of* W *over* R, i.e.,

$$\operatorname{vcd}_R W = \sup\{i \mid H^i(W; RW) \neq 0\}.$$

Remark.

$$\begin{aligned} \dim_R \partial\Sigma(W,S) &= \operatorname{vcd}_R W - 1 \\ &= \sup\{i \mid H^i(W; RW) \neq 0\} - 1 \\ &= \sup\{i \mid \check{H}^i(\partial\Sigma(W,S); R) \neq 0\} \end{aligned}$$

Theorem (Davis 1998). For a Coxeter system (W, S),

$$
\begin{aligned}
H^*(W; RW) &\big(\cong H_c^*(\Sigma(W,S); R) \cong \check{H}^{*-1}(\partial\Sigma(W,S); R)\big) \\
&\cong \bigoplus_{\substack{T \subset S \\ W_T \text{ is finite}}} \left(\mathbb{Z}(W^T) \otimes \tilde{H}^{*-1}(L(W_{S\setminus T}, S \setminus T); R)\right),
\end{aligned}
$$

where $\tilde{H}^*$ denotes the reduced cohomology and $\mathbb{Z}(W^T)$ is the free abelian group on W^T.

For each $w \in W$,

$$S(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S,

$$W^T := \{w \in W \mid S(w) = T\}.$$

Theorem (H 2001). For a Coxeter system (W, S),

$$H^*(W; RW)\big(\cong H_c^*(\Sigma(W,S); R) \cong \check{H}^{*-1}(\partial\Sigma(W,S); R)\big)$$
$$\cong \tilde{H}^{*-1}(L(W_{\tilde{S}}, \tilde{S}); R)$$
$$\oplus \left(\bigoplus_{\substack{T \subset S \\ W_T \text{ is finite} \\ S \setminus \tilde{S} \subsetneq T}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L(W_{S\setminus T}, S \setminus T); R) \right).$$

Definition. A Coxeter system (W, S) is said to be *irreducible* if, for any nonempty and proper subset T of S, $W \neq W_T \times W_{S\setminus T}$.

Let (W, S) be a Coxeter system. There exists a unique decomposition $\{S_1, \dots, S_r\}$ of S such that $W = W_{S_1} \times \cdots \times W_{S_r}$ and each Coxeter system (W_{S_i}, S_i) is irreducible. Let

$$\tilde{S} := \bigcup \{S_i \mid W_{S_i} \text{ is infinite}\}.$$

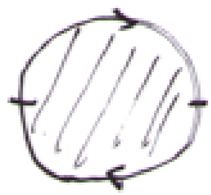
Then $W = W_{\tilde{S}} \times W_{S\setminus\tilde{S}}$ and $W_{S\setminus\tilde{S}}$ is finite.

Remark.

$$\begin{aligned}\dim_R \partial\Sigma(W,S) &= \sup\{i \mid \check{H}^i(\partial\Sigma(W,S);R) \neq 0\} \\ &= \sup\{i \mid \tilde{H}^i(L(W_{S\setminus T}, S\setminus T);R) \neq 0 \\ &\qquad\qquad \text{for some } T \subset S \text{ such that } W_T \text{ is finite}\}\end{aligned}$$

For every finite simplicial complex K, there exists a right-angled Coxeter system (W, S) such that $L(W, S) = \mathrm{sd}\ K$.

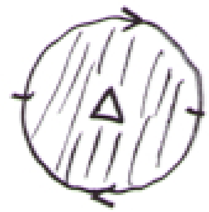
Example. $K =$ the projective plane. Then



$$\tilde{H}^i(L(W,S)) \cong \begin{cases} \mathbb{Z}_2, & i = 2 \\ 0, & i \neq 2 \end{cases}$$

Now $\tilde{S} = S$. For each $T \subset S$ such that $T \neq \emptyset$ and W_T is finite, $L(W_{S \setminus T}, S \setminus T)$ has the same homotopy type as the circle. Hence,

$$\tilde{H}^i(L(W_{S\setminus T}, S \setminus T)) \cong \begin{cases} \mathbb{Z}, & i = 1 \\ 0, & i \neq 1 \end{cases}$$



Therefore,

$$\check{H}^i(\partial\Sigma(W,S)) \cong \begin{cases} \mathbb{Z}_2, & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots, & i = 1, \\ 0, & \text{otherwise} \end{cases}$$

Thus $\dim \partial\Sigma(W,S) = 2$ and $\dim_{\mathbb{Q}} \partial\Sigma(W,S) = 1$.

The following problems are open.

Problem (Dranishnikov)

For each $3 \leq n \in \mathbb{N}$, does there exist a Coxeter system (W, S) such that $\dim \partial\Sigma(W, S) = n$ and $\dim_{\mathbb{Q}} \partial\Sigma(W, S) = 1$?

Problem

For each $3 \leq n \in \mathbb{N}$, does there exist a *right-angled* Coxeter system (W, S) such that $\dim \partial\Sigma(W, S) = n$ and $\dim_{\mathbb{Q}} \partial\Sigma(W, S) = 1$?

The above problem is equivalent to the following:

Problem

For each $3 \leq n \in \mathbb{N}$, does there exist a flag complex L such that

$$\max\{i \,|\, \tilde{H}^i(L - \sigma) \neq 0 \text{ for some } \sigma \in L \cup \{\emptyset\}\} = n$$

and

$$\max\{i \,|\, \tilde{H}^i(L - \sigma; \mathbb{Q}) \neq 0 \text{ for some } \sigma \in L \cup \{\emptyset\}\} = 1?$$

On cohomology

For a flag complex L which is not a cone,

$$\check{H}^i(\partial\Sigma(W,S)) \cong \tilde{H}^i(L) \oplus \left(\bigoplus_{\sigma \in L} \bigoplus_{\mathbb{Z}} \tilde{H}^i(L - \sigma) \right),$$

where (W,S) is a right-angled Coxeter system such that $L(W,S) = L$.