

BOUNDS ON BOUNDARY SLOPES FOR KNOTS

KAZUHIRO ICHIHARA

ABSTRACT. In this research note, a number of bounds on boundary slopes of essential surfaces embedded or immersed in 3-manifolds are presented. Also reports on computer-aided experiments, concerning embedded boundary slopes for Montesinos knots, are included.

1. INTRODUCTION

The aim of this article is to give a summary of the results, concerning boundary slopes of essential surfaces embedded or immersed in compact 3-manifolds with single torus boundary, obtained by the author and by joint works of the author with Shigeru Mizushima (Tokyo Institute of Technology). Also we include reports on computer-aided experiments, which gave us motivations for our projects.

We start with fundamental notations and definitions, which will be used throughout the article. Please see [11, 18] as basic references. The isotopy class of a nontrivial, unoriented simple closed curve on a torus T is called a *slope* on T . Recall that, on the peripheral torus of a *knot*, i.e., embedded simple closed curve, in the three-dimensional sphere S^3 , the set of non-meridional slopes is identified with the set of rational numbers. See [21] for example. Throughout the following, let M denote a compact, connected, orientable 3-manifold with single torus boundary ∂M .

The author would like to thank Shigeru Mizushima for allowing to include the results of our joint works and the reports on our experiments in this article. The author was partially supported by the Institute for Industrial Research of Osaka Sangyo University.

2. EMBEDDED BOUNDARY SLOPES FOR MONTESINOS KNOTS

A compact, connected, possibly non-orientable surface embedded in M is called *essential* if it is incompressible and boundary-incompressible. The concept of embedded

Date: February 15, 2006.

2000 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. boundary slopes, diameter, Montesinos knot.

essential surface was introduced mainly by Haken. They have played and are still playing important roles in the study of knots and 3-manifolds. The boundary of an embedded essential surface consists of a parallel family of non-trivial simple closed curves on ∂M . Thus they determine a slope on ∂M . The slope so determined is called the *embedded boundary slope* of the surface. The boundary slopes have been used to study the amount of essential surfaces in a sense. Note that it is also related to the study of exceptional Dehn surgery. Please see [6] for a survey.

Following the pioneering work obtained by Thurston [22], which determines the embedded boundary slopes for the figure-eight knot in S^3 , embedded boundary slopes are intensively studied for some class of knots in S^3 . First, for two-bridge knots, Hatcher and Thurston gave a complete description of all embedded boundary slopes [10]. Generalizing their method, Hatcher and Oertel gave in [9] an algorithm to list all embedded boundary slopes up for *Montesinos knots*. A Montesinos knot is defined as a knot obtained by composing a number of rational tangles in line.

By using Hatcher-Oertel's algorithm, the author and Mizushima studied numerical properties of embedded boundary slopes for Montesinos knots in [14]. Actually the following bounds were obtained. In the following, we are assuming for Montesinos knots that the number of tangles is at least three and each rational tangle is non-integral.

Theorem 2.1. *Let χ be the Euler characteristic of an embedded essential surface in a Montesinos knot exterior with non-meridional embedded boundary slope R , and $\#b$ the number of boundary components of the surface. Then, except for certain boundary slopes, the denominator Q of R is bounded as*

$$Q \leq \frac{-\chi}{\#b}.$$

The exceptions only occur from $(-2, 3, t)$ -pretzel knots for odd $t \geq 3$ or their mirror images. Exceptional boundary slopes for the knot satisfy a weaker bound

$$\frac{-\chi}{\#b} < Q \leq \frac{-\chi}{\#b} + 1,$$

while a stronger condition $\#b \geq 2$ holds in these cases.

A number of corollaries can be implied from this theorem. In particular, we can prove that the well-known Cabling Conjecture is true for Montesinos knots. Though this has been already achieved in [5] as a corollary of the result for strongly invertible knots, our method gives a direct proof for Montesinos knots.

Theorem 2.2. *Let χ_i be the Euler characteristic of an embedded essential surface in a Montesinos knot exterior with non-meridional embedded boundary slope R_i , and $\sharp s_i$ its number of sheets, for $i = 1, 2$ respectively. Then the difference $|R_1 - R_2|$ between the boundary slopes R_1 and R_2 is bounded as*

$$|R_1 - R_2| \leq 2 \left(\frac{-\chi_1}{\sharp s_1} + \frac{-\chi_2}{\sharp s_2} \right) + 4.$$

Here, following [10], we call the number of pieces of the surface in a small neighborhood of a point on a knot *the number of sheets*.

3. DIAMETER OF EMBEDDED BOUNDARY SLOPE SET

In [8], Hatcher showed that there are only finitely many embedded boundary slopes on ∂M . Moreover, by Culler-Shalen [2], it was shown that there are at least two embedded boundary slopes if M is the exterior of a non-trivial knot in S^3 . Therefore, we get the *embedded boundary slope set* for a knot as an invariant, which is regarded as a non-empty, finite subset of rational numbers. In view of this, Culler and Shalen considered the *diameter* of the embedded boundary slope set for a knot, defined as the difference between the greatest and the least. They actually proved that it is greater than two for a non-trivial knot in S^3 in [2]. In [15], by also using the Hatcher-Oertel's algorithm, we proved for a Montesinos knot that the diameter of the embedded boundary slope set is bounded from above by the twice of the minimal crossing number.

Theorem 3.1. *Let K be a Montesinos knot. Then, we have*

$$\mathcal{D}(K) \leq 2 \text{cr}(K),$$

where $\text{cr}(K)$ denotes the minimal crossing number of K and $\mathcal{D}(K)$ the diameter of the embedded boundary slope set for K . The equality holds for alternating Montesinos knots.

The related result for two-bridge knots is obtained in [20].

Also we obtained a lower bound on the diameter of the embedded boundary slope set for Montesinos knots in terms of some topological quantities of the surfaces which realize the greatest and the least boundary slopes. This will appear in the forthcoming paper [16].

Theorem 3.2. *Let K be a Montesinos knot, and R_1, R_2 be the greatest and the least in the embedded boundary slope set for K respectively. Then there exist embedded essential surfaces with boundary slopes R_1, R_2 such that the diameter $\mathcal{D}(K)$ of the embedded boundary slope set for K is bounded as*

$$\mathcal{D}(K) \geq 2\left(\frac{-\chi_1}{\#s_1} + \frac{-\chi_2}{\#s_2}\right),$$

where χ_i and $\#s_i$ denote the Euler characteristic and the number of sheets of the surface for $i = 1, 2$ respectively.

In [17], Ishikawa, Mattman and Shimokawa gave a lower bound on the diameter of the embedded boundary slope set for a knot by generalizing the Culler-Shalen's method. Their bound is described in terms of the Culler-Shalen norm of the embedded boundary slopes together with some additional data. Please see [17] about the definition of the Culler-Shalen norm for example. The following theorem gives an extension to their results. It will be included in the forthcoming paper [13]. Here a knot in S^3 is called *hyperbolic* if its complement admits a complete hyperbolic metric with finite volume.

Theorem 3.3. *For a hyperbolic knot K in S^3 , let R_1, \dots, R_n be the embedded boundary slopes and m the meridian. Suppose that m is not a strict boundary slope. Then we have*

$$2 \max_{1 \leq i \leq n} \left\{ \frac{\|R_i\|}{\|m\| \cdot \Delta(R_i, m)} \right\} \geq \mathcal{D}(K) \geq \max_{1 \leq i \leq n} \left\{ \frac{\|R_i\|}{\|m\| \cdot \Delta(R_i, m)} \right\}.$$

4. DISTANCE BETWEEN IMMERSED BOUNDARY SLOPES

In his famous lecture note [22], Thurston gave a concise construction of infinitely many 3-manifolds without embedded essential surfaces, and demonstrated that such 3-manifolds can appear frequently than expected. After that, a number of concepts have been developed for a substitute of embedded essential surface. Among them, immersed essential

surfaces are studied in detail since they seem to have important properties like embedded ones. Here by an *immersed surface* in M , we mean the image of a proper immersion of a compact surface into M . An immersed surface is said to be *essential* if it is π_1 -injective and relative π_1 -injective. Also, in this article, we will always assume that any immersed essential surface has embedded boundary, i.e., the restriction of the immersion to the boundary of the source surface is an embedding. Then the boundary again determines a slope on ∂M , called the *immersed boundary slope* of the surface.

Though embedded boundary slopes are only finitely many, immersed boundary slopes can exist infinitely many. For example, Maher showed in [19] that for any hyperbolic two-bridge knots, all slopes can be realized as immersed boundary slopes. Also Bart [1] showed such a phenomenon occurs for some hyperbolic arithmetic 3-manifolds.

On the other hand, Hass, Rubinstein and Wang proved in [7] that there are only finitely many immersed boundary slopes realized by the surfaces with bounded genera. In fact, they gave for hyperbolic 3-manifolds an explicit bound on the distance of two immersed boundary slopes in terms of the genera of the surfaces realizing the slopes. We give partial extensions to this result as follows, which will be included in the forthcoming paper [12]. In the following, for brevity, the boundary and the Euler characteristic of an immersed surface stand for the corresponding those of the source surface respectively.

Theorem 4.1. *Let χ_i be the Euler characteristic of an immersed essential surface in a hyperbolic knot exterior in S^3 with non-meridional immersed boundary slope R_i , and $\#s_i$ its number of sheets, for $i = 1, 2$ respectively. Then the difference $|R_1 - R_2|$ between the boundary slopes R_1 and R_2 is bounded as*

$$|R_1 - R_2| \leq 6 \left(\frac{-\chi_1}{\#s_1} + \frac{-\chi_2}{\#s_2} \right).$$

This inequality holds for any hyperbolic 3-manifold and for any meridian-longitude system on ∂M .

In the next, we used the distance $\Delta(R_1, R_2)$ of slopes R_1 and R_2 instead of the difference $|R_1 - R_2|$. The *distance* $\Delta(R_1, R_2)$ of slopes R_1 and R_2 is defined as the minimal geometric intersection number of the representatives of R_1 and R_2 . By definition, we see that $|R_1 - R_2| \leq \Delta(R_1, R_2)$ always holds, and the equality holds if R_1, R_2 are integral slopes.

Theorem 4.2. *Let χ_i be the Euler characteristic of an immersed essential surface in a torus knot exterior in S^3 with non-meridional immersed boundary slope R_i , and $\#b_i$ its number of sheets, for $i = 1, 2$ respectively. Then the distance $\Delta(R_1, R_2)$ between the boundary slopes R_1 and R_2 is bounded as*

$$\left(\frac{-\chi_1}{\#b_1} + \frac{-\chi_2}{\#b_2}\right) + 2 \leq \Delta(R_1, R_2) \leq 2\left(\frac{-\chi_1}{\#b_1} + \frac{-\chi_2}{\#b_2}\right) + 4.$$

This inequality holds for all small Seifert fibered spaces.

Theorem 4.3. *Let χ_i be the Euler characteristic of an immersed spanning surface without triple points for a knot in S^3 with boundary slope R_i for $i = 1, 2$ respectively. Then the difference $|R_1 - R_2|$ between the boundary slopes R_1 and R_2 is bounded as*

$$|R_1 - R_2| \leq 2\left(\frac{-\chi_1}{\#s_1} + \frac{-\chi_2}{\#s_2}\right) + 4.$$

To prove the last theorem, we use so-called Whitney-Massey's Theorem, which concerns two-dimensional knots in 4-space. Thus it actually holds for such surfaces even if they are not essential.

5. EXPERIMENTS FOR BOUNDARY SLOPES

In this section, we report computer-aided experiments we did, which gave us motivations of the results obtained in [14]. We used the program developed by Dunfield [4], which implements the algorithms by Hatcher and Thurston developed in [10] and by Hatcher and Oertel developed in [9].

5.1. Range of knots. In the experiments, we considered the knots of the following ranges. Please note that we did not any treatment to exclude non-hyperbolic knots.

5.1.1. Montesinos knot. We consider the Montesinos knots corresponding to the triples of irreducible fractions $(p_1/q_1, p_2/q_2, p_3/q_3)$, where each p_i/q_i satisfies $-7 \leq p_i \leq 7$, $p_i \neq 0$, $2 \leq q_i \leq 7$. Please note that Dunfield's program does not accept the triples corresponding to links. Thus we have the following table:

number of triples of fractions	175616 (= 56^3)
irreducible ones corresponding to knots	95952

5.1.2. *pretzel knot*. We consider the triples of irreducible fractions $(p_1/q_1, p_2/q_2, p_3/q_3)$, where each p_i/q_i satisfies $p_i = \pm 1$, $2 \leq q_i \leq 20$. These correspond to so-called *pretzel knots*. The number of the knots we considered is;

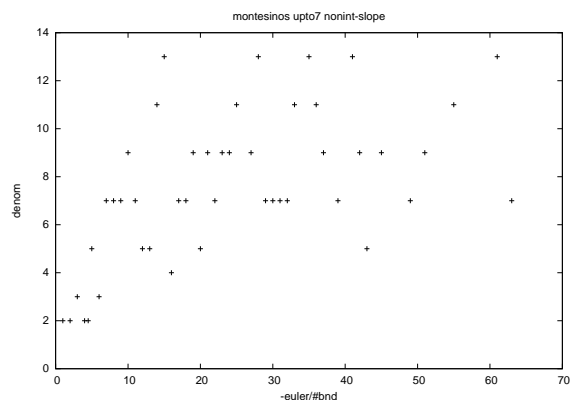
number of triples of fractions	54872 (=38 ³)
irreducible ones corresponding to knots	25272

5.1.3. *two-bridge knot*. We consider irreducible fractions p/q , where each p/q satisfies $2 \leq p \leq 100$, $p + 2 \leq q \leq 100$, and q is odd. By taking the closure of each corresponding rational tangle, we have a so-called *two-bridge knot*. The number of such knots we considered is 1908.

5.2. **denominator of boundary slope**. For brevity, denote by v the ratio of the negative of the Euler characteristic and the number of boundary components of an essential surface having a boundary slope with denominator d . In this subsection, we report on the experiments concerning the relationship between v and d for non-integral boundary slopes for knots. In processing of the obtained data, we found the maximum d among the boundary slopes with the same v , and based on this, we obtained a graph on the v - d plane. We thus remark that not all the points are plotted in the graph.

The following table and graph were obtained for Montesinos knots in the range:

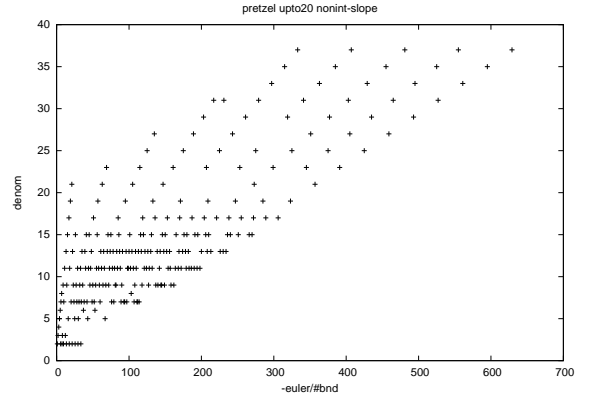
number of triples of fractions	175616
irreducible ones corresponding to knots	95952
the number of boundary slopes	1224222
the number of non-integral boundary slopes	30624



We could observe certain linear relationship between them. In this experiment, the maximum of the value d/v is 2, and it is realized by the Montesinos knot $K(-7/3, -1/7, 5/2)$ for example: For this knot, we find a boundary slope with $v = 1$, $d = 2$.

We obtained the following table and graph for pretzel knots in the range. We could observe certain linear relationship between them more clearly.

number of triples of fractions	54872
irreducible ones corresponding to knots	25272
the number of boundary slopes	146532
the number of non-integral boundary slopes	17232

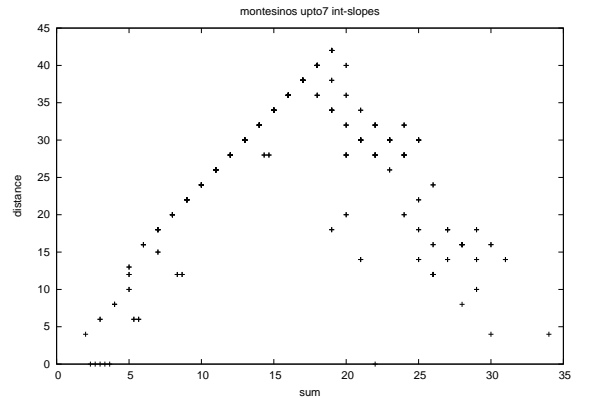


In this experiment, the maximum of the value d/v is also 2, and it is realized by the pretzel knot $P(-2, 3, 7)$ for example: For this knot, we find a boundary slope $37/2$ with $v = 1$, $d = 2$.

5.3. distance between integral slopes. For brevity, denote by v the ratio of the negative of the Euler characteristic and the number of boundary components of an essential surface with a boundary slope. In this subsection, we report on the experiments concerning the relationship between the sums $v_1 + v_2$ and the distances Δ for pairs of integral boundary slopes. In processing of the obtained data, we find the maximum Δ among the boundary slopes with the same $v_1 + v_2$, and based on this, we obtained a graph. We thus again remark that not all the points for pairs are plotted in the graph.

The following table and graph were obtained for Montesinos knots in the range:

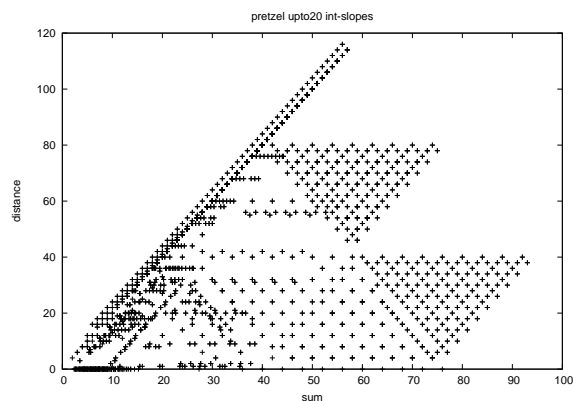
number of triples of fractions	175616
irreducible ones corresponding to knots	95952
the number of pairs of boundary slopes	10094494
the number of pairs of integral boundary slopes	9750454



We could observe certain linear relationship between them.

For pretzel knots in the range, the following table and graph were obtained:

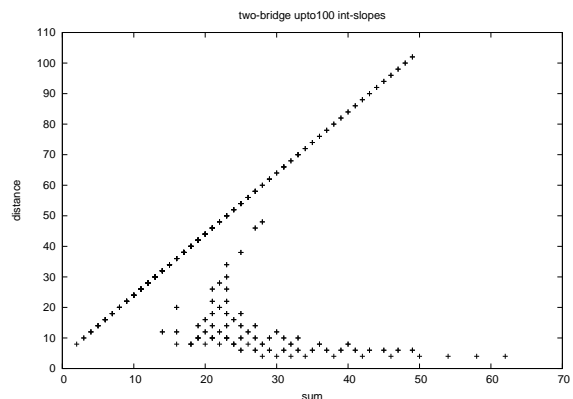
number of triples of fractions	54872
irreducible ones corresponding to knots	25272
the number of pairs of boundary slopes	513048
the number of pairs of integral boundary slopes	410556



We could observe certain linear relationship between them more clearly.

We also have the following table and graph for two-bridge knots in the range. Remark that all boundary slopes for two-bridge knots are integral.

number of fractions	1908
the number of pairs of boundary slopes	30124

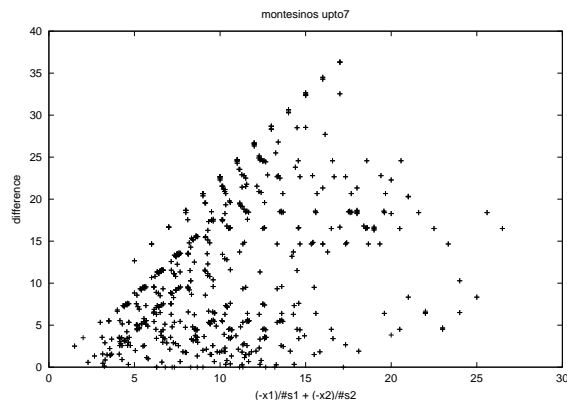


We could observe rather clear linear relationship between them.

5.4. difference between non-integral slopes. Let δ be the difference $|p_1/q_1 - p_2/q_2|$ of two boundary slopes $p_1/q_1, p_2/q_2$. For an embedded essential surface with boundary slope p_i/q_i , let χ_i be its Euler characteristic and $\#s_i$ the number of sheets for the surface, respectively for $i = 1, 2$. In this subsection, we report on the experiments concerning the relationship between the values $(-\chi_1/\#s_1) + (-\chi_2/\#s_2)$ and differences δ for the pairs of boundary slopes, at least one of which is non-integral. In processing of the obtained data, we did certain simplifications, and so, remark that not all the points for pairs are plotted in the graph.

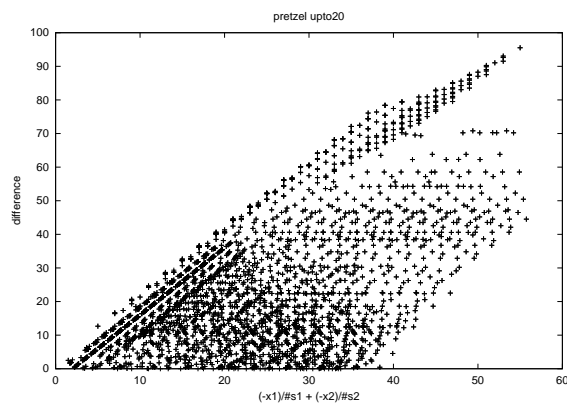
The following table and graph were obtained for Montesinos knots in the range:

number of triples of fractions	175616
irreducible ones corresponding to knots	95952
the number of pairs of boundary slopes	10094494
the number of pairs of such boundary slopes	344040



For pretzel knots in the range, we have the following table and graph:

number of triples of fractions	54872
irreducible ones corresponding to knots	25272
the number of pairs of boundary slopes	513048
the number of pairs of such boundary slopes	102492



From these, we could observe certain linear relationship between them.

REFERENCES

1. A. Bart, [2004], “Immersed and virtually embedded boundary slopes in arithmetic manifolds”, *J. Knot Theory Ramifications*, vol. 13, no.5, pp. 587–596.
2. M. Culler and P.B. Shalen, [1984], “Bounded, separating, incompressible surfaces in knot manifolds”, *Invent. Math.*, vol. 75, pp. 537–545.
3. M. Culler and P.B. Shalen, [1999], “Boundary slopes of knots”, *Comment. Math. Helv.*, vol. 74, pp. 530–547.
4. N. Dunfield, a computer program freely available from <http://www.its.caltech.edu/~dunfield/montesinos/index.html>
5. M. Eudave-Muñoz, [1992], “Band sums of links which yield composite links. The cabling conjecture for strongly invertible knots”, *Trans. Amer. Math. Soc.*, vol. 330, pp. 463–501.
6. C.McA. Gordon, [1999], “Small surfaces and Dehn filling”, *Geometry and Topology Monographs*, vol. 2, Proceedings of the Kirbyfest, pp. 177–199.
7. J. Hass, J.Hyam. Rubinstein and S.-C. Wang, [1999], “Boundary slopes of immersed surfaces in 3-manifolds”, *J. Differential Geom.*, vol. 52, no. 2, pp. 303–325.

8. A. Hatcher, [1982], “On the boundary curves of incompressible surfaces”, *Pacific J. Math.*, vol. 99, pp. 373–377.
9. A. Hatcher and U. Oertel, [1989], “Boundary slopes for Montesinos knots”, *Topology*, vol. 28, no. 4, pp. 453–480.
10. A. Hatcher and W.P. Thurston, [1985], “Incompressible surfaces in 2-bridge knot complements”, *Invent. Math.*, vol. 79, pp. 225–246.
11. J. Hempel, [1976], *3-Manifolds*, Annals of Math. Studies vol. 86, Princeton.
12. K. Ichihara, “Distances between immersed boundary slopes”, in preparation.
13. K. Ichihara, “Euclidean length on a horotorus and the Culler-Shalen norm of slopes”, in preparation.
14. K. Ichihara and S. Mizushima, “Bounds on numerical boundary slopes for Montesinos knots”, preprint available at arXiv:math.GT/0503190.
15. K. Ichihara and S. Mizushima, “Crossing number and diameter of boundary slope set of Montesinos knot”, preprint available at arXiv:math.GT/0510370.
16. K. Ichihara and S. Mizushima, “A lower bound of the boundary slope diameter for Montesinos knot”, in preparation.
17. M. Ishikawa, T.W. Mattman and K. Shimokawa, “Exceptional surgery and boundary slopes”, to appear in *Osaka J. Math.*, also available at arXiv:math.GT/0211147.
18. W. Jaco, [1980], *Lectures on three-manifold topology*, Conf. Board of Math. Sci., vol. 43, Amer. Math. Soc.
19. J. Maher, [1999], “Virtually embedded boundary slopes”, *Topology Appl.*, vol. 95, no. 1, pp. 63–74.
20. T.W. Mattman, G. Maybrun and K. Robinson, “Boundary slope diameter and crossing number of 2-bridge knots”, preprint available at arXiv:math.GT/0510496.
21. D. Rolfsen, [1976], *Knots and Links*, Publish or Perish, Berkeley, California.
22. W.P. Thurston, [1978], *The geometry and topology of 3-manifolds*, Lecture notes, Princeton University.

COLLEGE OF GENERAL EDUCATION, OSAKA SANGYO UNIVERSITY, 3-1-1 NAKAGAITO, DAITO, OSAKA 574-8530, JAPAN

E-mail address: ichihara@las.osaka-sandai.ac.jp