

# SUM OF THE EDGE LENGTHS OF A GEODESIC GRAPH

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## ABSTRACT

Consider an embedding of a complete graph of order four into the 2-sphere such that each edge becomes a shortest geodesic connecting its endpoints. Then we show that the sum of the edge lengths is at most  $4\pi$ , and is bigger than  $3\pi$  if the graph is not contained in any hemisphere.

## はじめに

2002年12月に京都大学数理解析研究所で行われた研究集会「双曲空間に関連する研究とその展望」では、「Area of a cellular complex in a hyperbolic manifold」という題目で、双曲多様体に測地的に埋め込まれた2次元胞体複体の面積に関して発表しました。本稿では、その主定理の証明において鍵となった、Guddumの定理 [1] の拡張を与えます。尚、研究集会で発表した結果に関しては、プレプリント [3] を参照下さい。

## 1. RESULTS

In this article, we consider a finite graph geodesically embedded into a surface with constant curvature metric, and estimate the sum of the edge lengths. As usual, we regard a finite graph as a 1-dimensional cellular complex by setting a vertex as a 0-cell and an edge as a closed 1-cell. Given an embedding  $f$  of a finite graph into a surface, its image  $G$  is obviously identified with the original graph. Thus we say that the image of a vertex

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and an edge under  $f$  a *vertex* and an *edge* of  $G$ . By  $S^2$ , we mean the two dimensional sphere endowed with the Riemannian metric of constant curvature  $+1$ . Then our result is the following.

**Theorem 1.** *Let  $G$  be the image of an embedding of the complete graph  $K_4$  of order 4 into  $S^2$ . Suppose that*

- (1) *each edge of  $G$  is a shortest geodesic arc on  $S^2$  connecting its endpoints, and*
- (2)  *$G$  is not contained in any hemisphere of  $S^2$ .*

*Let  $E$  be the sum of the length of the edges of  $G$ . Then  $3\pi < E \leq 4\pi$  holds.*

This almost follows from the result of Guddum [1]. In fact, his theorem in [1] implies the inequality  $3\pi \leq E \leq 4\pi$ . In the next section, we will prove the theorem above using purely elementary spherical geometry.

A generalization of this estimate to the case of any graph embedded in  $n$ -sphere  $S^n$  will appear in [4]. Our estimate depends upon the combinatorics of the graph only.

Here we append an easy observation for more general cases. Let  $F_g$  be a closed, orientable surface of genus  $g \geq 2$  with a fixed Riemannian metric of constant curvature  $-1$ . For convenience, let  $F_0$  denote  $S^2$ .

**Proposition 1.** *Let  $G$  be the image of an embedding of a graph into  $F_g$  where  $g \neq 1$ . Suppose that*

- (1) *each edge  $e$  of  $G$  is a shortest geodesic arc on  $F_g$  connecting its endpoints, and*
- (2) *the closure of each component of  $F_g - G$  is a convex polygon on  $F_g$ .*

*Then the sum of the length of the edges of  $G$  is greater than  $\pi|2 - 2g|$ .*

*Proof.* Let  $\sigma$  be a complementary face of  $G$ , i.e., the closure of a component of  $F_g - G$ . By  $Area(\sigma)$  and  $Length(\partial\sigma)$ , we denote the area of  $\sigma$  and the length of the boundary  $\partial\sigma$  of  $\sigma$  respectively. We consider the ratio  $Area(\sigma)/Length(\partial\sigma)$ . This ratio is strictly less than the corresponding ratio for the disk on  $F_g$  which has equilong boundary as  $\sigma$ . See [2] for a survey. By elementary calculations, the ratio for such a disk is shown

to be less than 1 for any  $g \neq 1$ . This implies  $Length(\partial\sigma) > Area(\sigma)$  holds. By summing the inequalities up over all complementary faces, we have  $\sum Length(\partial\sigma) > \sum Area(\sigma) = 2\pi|2 - 2g|$ , where the last equality follows from the Gauss-Bonne's theorem. Then the sum of the length of the edges of  $G$ , which is equal to the half of  $\sum Length(\partial\sigma)$ , is greater than  $\pi|2 - 2g|$ .  $\square$

## 2. PROOF

Let us start with recalling fundamentals of spherical geometry. Let  $u_1, u_2, u_3$  be points on  $S^2$  such that no two of them are antipodal and no great circle includes all the three points. Let  $\Lambda_i$  be the closed hemisphere whose boundary contains the other two points than  $u_i$  and whose interior contains  $u_i$  for  $i = 1, 2, 3$ . The *spherical triangle*  $\Delta$  with the vertices  $u_1, u_2, u_3$  is defined as the intersection  $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ . Then we have the following:

- $\Delta$  is convex, i.e., any pair of points in  $\Delta$  is connected by a geodesic arc in  $\Delta$ . Moreover the arc is shortest among the arcs connecting the points, and the length is equal to the spherical distance between the points which is strictly less than  $\pi$ .
- The length of an edge of  $\Delta$  is less than the sum of the length of the other two edges (*the triangle inequality*).

*Proof of Theorem 1.* Let  $v_1, v_2, v_3, v_4$  be the vertices of  $G$ . Let  $e_{ij}$  denote the edge of  $G$  connecting  $v_i$  and  $v_j$  for  $1 \leq i, j \leq 4$ . Note that the assumption (1) implies that the length of  $e_{ij}$  is equal to the spherical distance  $d_{ij}$  on  $S^2$  between  $v_i$  and  $v_j$  for  $1 \leq i, j \leq 4$ . Thus it suffice to show that

$$3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi .$$

In the following, the antipodal point of  $v_i$  is denoted by  $v_{i+4}$  for  $1 \leq i \leq 4$ . Also  $d_{ij}$  denotes the spherical distance between  $v_i$  and  $v_j$  for  $1 \leq i, j \leq 8$ .

First we consider the case that a couple of the vertices, say  $v_1$  and  $v_2$ , are antipodal, equivalently,  $d_{12} = \pi$ . This implies that  $d_{13} + d_{32} = d_{14} + d_{42} = \pi$  holds. Together with  $0 < d_{34} \leq \pi$ , we have  $3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi$ .

Next consider the case that all the four vertices are contained in a great circle. Suppose for example that  $v_1, v_2, v_3, v_4$  lies in a great circle  $\Gamma$  in this

order. Since  $G$  is the image of an embedding, the edge  $e_{13}$  is not contained in  $\Gamma$ . This implies that  $e_{13}$  is a half of a great circle and  $d_{13} = \pi$ . Also we see that  $d_{24} = \pi$  and so we obtain  $\sum_{1 \leq i < j \leq 4} d_{ij} = 4\pi$ .

Thus, in the following, we assume that  $d_{ij} \neq \pi$  for  $1 \leq i, j \leq 4$  and at most three vertices of  $G$  lie on a great circle.

Next consider the case that three vertices are contained in a great circle. Suppose for example that  $v_1, v_2$  and  $v_3$  lie on a great circle. Then, by the triangle inequality, we have  $d_{41} + d_{42} > d_{12}$ ,  $d_{42} + d_{43} > d_{23}$  and  $d_{43} + d_{41} > d_{31}$ . These are added to obtain

$$2(d_{41} + d_{42} + d_{43}) > d_{12} + d_{23} + d_{31} = 2\pi .$$

Thus

$$\sum_{1 \leq i < j \leq 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} > \pi + 2\pi = 3\pi .$$

In the same way as above, we have  $d_{45} + d_{46} + d_{47} > \pi$ . Since  $d_{4j} = \pi - d_{4(j+4)}$  for  $j = 1, 2, 3$ ,

$$\begin{aligned} \sum_{1 \leq i < j \leq 4} d_{ij} &= (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &= 3\pi - (d_{45} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &< 3\pi - \pi + 2\pi = 4\pi \end{aligned}$$

holds.

Finally we consider the case that the four vertices are in a general position: We assume that  $d_{ij} \neq \pi$  for  $1 \leq i, j \leq 4$  and at most two vertices of  $G$  lie on a great circle. This means that for any three of the points there is a triangular face which includes the three points as vertices.

Then, by the triangle inequality, we have  $d_{53} + d_{63} > d_{56}$ ,  $d_{54} + d_{64} > d_{56}$ ,  $d_{53} + d_{54} > d_{34}$  and  $d_{63} + d_{64} > d_{34}$ . Add these to obtain

$$d_{53} + d_{63} + d_{54} + d_{64} > d_{34} + d_{56} .$$

Here note that  $d_{ij} = \pi - d_{(i-4)j}$  for  $i = 5, 6$ ,  $j = 1, 2, 3$ , and  $d_{56} = d_{12}$ . These imply that

$$4\pi - (d_{13} + d_{23} + d_{14} + d_{24}) > d_{34} + d_{12} .$$

Consequently we have

$$4\pi > \sum_{1 \leq i < j \leq 4} d_{ij} .$$

In the following, let  $\Delta$  be the spherical triangle bounded by  $e_{12}$ ,  $e_{23}$  and  $e_{31}$ .

**Claim 1.** *The antipodal point  $v_8$  of  $v_4$  is included in the interior of  $\Delta$ .*

*Proof.* Let  $\Gamma_i$  be the great circle including an edge of  $\Delta$  but not including  $v_i$  for  $i = 1, 2, 3$ . By the assumption above,  $v_4$  and hence  $v_8$  never lie on  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ . Note that  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$  decomposes  $S^2$  into eight spherical triangles.

Assume for a contradiction that  $v_8$  is not included in the interior of  $\Delta$ . Then  $v_4$  is included in the interior of one of the seven spherical triangles other than the antipodal image of  $\Delta$ . This implies all the four points  $v_1$ ,  $v_2$ ,  $v_3$  and  $v_4$  are included in the closed hemisphere bounded by one of  $\Gamma_1$ ,  $\Gamma_2$  or  $\Gamma_3$ . Since the four vertices are assumed in a general position, there is a hemisphere which contains whole  $G$ . This contradicts the assumption (2) of the theorem.  $\square$

**Claim 2.** *The inequality  $d_{12} + d_{13} > d_{82} + d_{83}$  holds.*

*Proof.* Since the length of each edge is less than  $\pi$ , the edge  $e_{13}$  intersects the great circle including  $v_2$  and  $v_8$  at just one point  $v_9$ . Let  $d_{i9}$  or  $d_{9i}$  denote the distance between  $v_i$  and  $v_9$  for  $1 \leq i \leq 9$ . The distance  $d_{19}$  is realized by a geodesic arc included in  $e_{13}$  and also is  $d_{93}$ . Thus  $d_{13} = d_{19} + d_{93}$  holds.

The distance  $d_{29}$  is realized by a geodesic arc  $e_{29}$  in  $\Delta$  since  $\Delta$  is convex. In particular, the arc  $e_{29}$  contains  $v_8$  and so  $d_{29} = d_{28} + d_{89}$  holds.

Together with the triangle inequality  $d_{12} + d_{19} > d_{29}$  and  $d_{98} + d_{93} > d_{83}$ , we conclude

$$d_{12} + d_{13} = d_{12} + d_{19} + d_{93} > d_{29} + d_{93} = d_{28} + d_{89} + d_{93} > d_{28} + d_{83} .$$

$\square$

In the same way, we have  $d_{21} + d_{23} > d_{81} + d_{83}$  and  $d_{31} + d_{32} > d_{81} + d_{82}$ . By adding these inequalities, we obtain

$$d_{12} + d_{23} + d_{31} > d_{81} + d_{82} + d_{83} .$$

Together with the equations  $d_{8j} = \pi - d_{4j}$  for  $i = 1, 2, 3$ , we conclude that

$$\sum_{1 \leq i < j \leq 3} d_{ij} > 3\pi - \sum_{1 \leq k \leq 3} d_{k4} .$$

This completes the proof. □

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