Essential surfaces in three-dimensional manifolds

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Contents

Introduction

The main subject of this thesis is the *essential surfaces* in 3-manifolds. An embedded or immersed surface in a 3-manifold is called *essential* if
(1) it does not come from a simply-connected surface,
(2) the embedding or immersion induces injective maps of the fundamental groups and of the sets of homotopy classes of proper arcs, and
(3) it is not freely homotopic into the boundary of the 3-manifold.

The essential surface has played a very important role in the study of 3-manifolds. For example, Waldhausen [**64**] proved that irreducible 3-manifolds with embedded essential surfaces have the topological rigidity; *a relative homotopy equivalence induces a homeomorphism.* Moreover, it was proved for such manifolds that Thurston's Geometrization Conjecture; *every compact orientable 3-manifold has a canonical decomposition by geometric pieces*, is true [**47, 59, 60, 61, 63**].

Throughout the thesis, we consider the questions; *how many essential surfaces exist in 3-manifolds* and *what properties do essential surfaces have, in particular, in hyperbolic 3-manifolds?* Being concerned with these questions, we study essential surfaces in four different situations, which correspond to Chapter 1, 2, 3 and 4.

Chapter 1: Boundary curves of essential surfaces.

To know how many essential surfaces exist in a 3-manifold with non-empty boundary, it seems useful to study the boundary curves of such surfaces. We set up the question: *Which closed curve on the boundary of a 3-manifold is contained in an immersed essential surface as a boundary component?*

An essential surface can be homotoped so that it does not intersect the compressing disks of the boundary of a 3-manifold. Hence, the focus of our interest is on irreducible 3-manifolds with incompressible boundary. The first step might be to study the question for simple pieces of the Jaco-Shalen-Johannson decomposition. In a *trivial or twisted I-bundle over a surface*, every non-trivial closed curve on the boundary is contained in a vertical annulus as a boundary component. On the other hand, it is known that the boundary curves of essential surfaces in *Seifert fibered 3-manifolds* are severely restricted. Detailed study was done in [**57**].

The remaining pieces are *acylindrical* ones, that is to say, the irreducible and ∂-irreducible 3-manifolds containing no essential tori and annuli. In this chapter, which is based on [**25**], we give explicit examples of acylindrical 3-manifolds having the property that every non-trivial closed curve on the boundary can appear as a boundary component of an immersed essential surface. The examples will be hyperbolic 3-manifolds with totally geodesic boundary, which are known to be acylindrical. We also show that the immersed surfaces in some particular class of our examples are virtually embedded, namely, it is lifted in a finite cover of the 3-manifold as an embedded one.

This result seems to indicate that there are *sufficiently many* immersed essential surfaces in 3-manifolds with non-empty boundary. In fact, the following conjecture is a motivation of this study, see [**5, 6**] for survey; *a closed irreducible 3-manifold with infinite fundamental group (A) contains an immersed essential surface, and (B) is finitely covered by a 3-manifold with embedded essential surfaces.* By recent works [**13, 16**], it is proved that (B) implies that Thurston's Geometrization Conjecture is true for such 3-manifolds.

On the other hand, recently, it is shown in [**18, 19**] that only finitely many *simple* closed curves on the boundary are realized as the boundaries of immersed essential surfaces of bounded genus for an acylindrical 3-manifold. Concerning these results, we show that only finitely many closed curves, up to homotopy, can appear as boundary components of essential surfaces with bounded Euler number on the boundary of an acylindrical 3-manifold.

Chapter 2: Framed link presentations of surface bundles.

A fiber surface in a 3-manifold fibering over the circle is a particular example of an embedded essential surface. It is well-known that every closed orientable 3-manifold has a framed link presentation, meaning that it is obtained by a framed surgery on some framed link in the 3-sphere [**33**]. Moreover such a framed link can be taken as a fibered link with some framing [**8**].

In this chapter, which is based on [**24**], we show that every closed orientable 3-manifold fibering over the circle is represented by a fibered link with the framings induced from the fibration of its complement.

Not every 3-manifold admits a fibration over the circle. However, Thurston conjectured [**61**] that every hyperbolic 3-manifold is finitely covered by a 3-manifold fibering over the circle. The first explicit example of a 3-manifold which contains no embedded essential surfaces and is finitely covered by a 3-manifold fibering over the circle is obtained in [**54**]. The result of this chapter is a byproduct of the study of this example.

The latter half of the thesis is based on the joint works [**26**] and [**27**] with Dr. Makoto Ozawa. In Chapter 3 and Chapter 4, we are mainly concerned with the geometric behavior of embedded essential surfaces in knot complements in the 3-sphere.

The classification of knots in the 3-sphere is reduced to the homeomorphism problem of the knot complements by the solution of a long standing Knot Complement Conjecture in [**17**]. Hence, essential surfaces must play an important role there, see [**22**] for example. Also many other problems in knot theory, in particular concerning Dehn surgery, are closely related with essential surfaces in their complements [**30**].

The geometric behavior of essential surfaces in hyperbolic 3-manifolds is of interest not only in Topology. An essential surface in a hyperbolic 3-manifold gives a surface subgroup in the holonomy image of the fundamental group of the manifold. Such subgroups are important subjects and have been well studied in Kleinian group theory.

Chapter 3: Knot complements without geodesic surfaces.

A large number of works have been done on essential surfaces in knot complements, in particular on the existence of embedded ones. See [**21, 36, 37, 38, 39, 42, 49**] for examples. On the geometric behavior of embedded essential surfaces in hyperbolic knot complements, it was conjectured [**43**] that the complements of knots in the 3-sphere have no closed embedded totally geodesic surfaces.

In this chapter, we show that the conjecture is true for hyperbolic 3-bridge knots and hyperbolic double torus knots in the 3-sphere. Besides, we give a few topological criteria for deciding whether a closed embedded essential surface in knot complements fails to be totally geodesic. Roughly speaking, we show that sufficiently "complicated" surfaces fail to be totally geodesic.

The conjecture seems to be meaningful only for knots in the 3-sphere. In fact, there is a *link* in the 3-sphere whose complement contains a closed embedded totally geodesic surface [**43**]. The complements of the figure-eight knot [**55**] and some other knots [**7**] in the 3-sphere contains *immersed* totally geodesic surfaces. It is known [**48**] that there are cusped hyperbolic 3-manifolds other than knot complements in the 3-sphere which contain closed embedded totally geodesic surfaces.

Chapter 4: Accidental surfaces in knot complements.

Consider an embedded essential surface in a 3-manifold with toral boundary. A non-trivial simple closed curve on the surface is called an *accidental peripheral* if there is an annulus running from the curve into the boundary of the 3-manifold. We call this annulus an *accidental annulus*, and such a surface containing an accidental peripheral an *accidental surface*. It is known for many knots in the 3-sphere that all closed essential surfaces in their exteriors are accidental.

In this chapter, we first show that all accidental annulus for an accidental surface determine the unique slope on the boundary of the knot exterior. Hence let us call this slope the *accidental slope* for the accidental surface. Moreover, in the case that this slope is not meridional, we show that all accidental peripherals are mutually isotopic on the surface.

Next, we give a characterization of accidental surfaces. As an application, we give a necessary and sufficient condition for knots to bound a totally knotted Seifert surface.

Finally, we give some criteria for checking whether a given closed essential surface is accidental or not. We also give some corollaries used to determine the accidental slope.

There are two motivations to study accidental surfaces.

One is concerned with the conjecture considered in the previous chapter. When the interior of the 3-manifold admits a hyperbolic metric of finite volume, Thurston showed [**60**] that a closed embedded essential surface is accidental if and only if corresponding surface subgroup is not quasi-Fuchsian. Hence accidental surfaces fail to be totally geodesic. It is still unsolved which knots in the 3-sphere have non-accidental surfaces in their exteriors.

The other is concerned with Dehn surgery. The question is whether a closed essential surface in a knot complement remains essential after Dehn surgeries on the knot. It is shown in [**12**, Theorem 2.4.3] that an accidental surface with an integral accidental slope γ remains essential after γ'-Dehn surgery if and only if the geometric intersection number of γ and γ' is just one. Concerning this result, it is a natural question how many accidental slopes exist for an accidental surface, and our result answers this question.

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