Longitudinal Seifert Fibered Surgeries on Hyperbolic Knots

Kazuhiro Ichihara*, Kimihiko Motegi** and Hyun-Jong Song

Abstract

Let K be a hyperbolic, fibered knot in the 3-sphere S^3. Then the exterior is regarded as a mapping torus of a compact, once punctured surface with a monodromy isotopic to a pseudo-Anosov automorphism. Performing a longitudinal surgery on K, we obtain a 3-manifold which is naturally regarded as the mapping torus of the capped off surface with the capped off monodromy; the dual knot (the core of the filled solid torus in the resulting 3-manifold) is a section for the surface bundle. Generically the resulting 3-manifold is still hyperbolic, in other words, the capped off monodromy is still isotopic to a pseudo-Anosov automorphism. Gabai found a hyperbolic, fibered knot in S^3 on which a longitudinal surgery produces a toroidal manifold, and now it is known that there are infinitely many such hyperbolic, fibered knots. On the other hand, there have been no known examples of hyperbolic, fibered knots in S^3 with longitudinal, Seifert fibered surgeries, and Teragaito asks if there are no such examples. We give an answer this question by constructing an infinite family of hyperbolic, fibered knots in S^3 each of which admits a longitudinal, Seifert fibered surgeries. Besides our examples show existence of boundary slopes which can be also Seifert fibered slopes.

We also give a condition assuring that the given section in a Seifert fibered, surface bundle over the circle is hyperbolic in terms of the "projection" in the fiber surface.

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1 Introduction

This paper consists of two parts. The problem we consider in the first part is motivated by [27], [28] asking existence of longitudinal Seifert fibered surgeries on hyperbolic knots in the 3-sphere. The problem in the second part was raised during our study about the first problem, which asks a necessary and sufficient condition for sections in a Seifert fibered, surface bundle over the circle being hyperbolic, and which is also related with a study of Nielsen-Thurston types of surface-automorphisms.

Let K be a hyperbolic, fibered knot in a closed 3-manifold M whose fiber surface has genus greater than one. Then its exterior $E(K)$ is regarded as a mapping torus of a once punctured, compact, orientable surface S with a monodromy map $h : S \to S$ which is isotopic to a pseudo-Anosov automorphism: $E(K) = (S \times [0,1])/\{(x,0) = (h(x),1)\}$. (For terminologies about surface-automorphisms, see [6], [8], [31].) The slope (i.e., an unoriented isotopy class of simple closed curves) on $\partial E(K)$ represented by a boundary of the fiber surface is called a *fiber slope*. Thus by performing a Dehn surgery along the fiber slope, we obtain a closed 3-manifold $M_f = (F \times [0,1])/\{(x,0) = (f(x),1)\}$, where F is the capped off closed surface obtained from the fiber surface S and $f : F \to F$ is the capped off monodromy obtained from the monodromy h. It follows that a surgery on K along the fiber slope creates a Seifert fiber space (resp. toroidal manifold) if and only if the capped off monodromy is isotopic to a periodic (resp. reducible) automorphism. Generically the resulting 3-manifold is still hyperbolic, in other words, the capped off monodromy is still isotopic to a pseudo-Anosov automorphism. In fact, if the monodromy h has a prong ≥ 2 singularity at the boundary, then the invariant measured singular foliation on S can be naturally extended to that of the capped off surface.

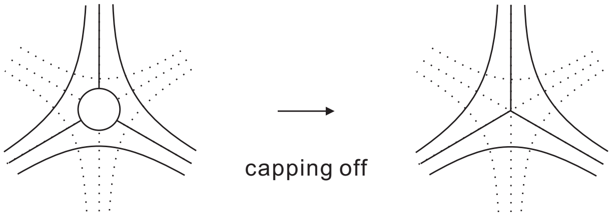

Figure 1: Pseudo-Anosov automorphism with prong = 3 singularity

Thus we have:

Proposition 1.1 *Let K be a hyperbolic, fibered knot with a monodromy isotopic to a pseudo-Anosov automorphism having a prong n singularity at the boundary. If $n \geq 2$, then the resulting manifold obtained by Dehn surgery along the fiber slope is hyperbolic, in particular it cannot be Seifert fibered.*

Assume that K is a fibered knot in S^3. Then, by definition, a fiber slope is a longitudinal slope, and a surgery along a fiber slope is often called a *longitudinal surgery*.

In [11, Remark 8.7(v)] Gabai observed that there is a hyperbolic, fibered knot in S^3 on which a longitudinal surgery yields a toroidal manifold by investigating the monodromy after Dehn filling using Penner's algorithm. Furthermore, Osoinach's construction in [23] shows that there are infinitely many such examples. Thus, it is natural to ask:

Question 1.2 *Can we obtain a Seifert fiber space by a longitudinal surgery on a hyperbolic, fibered knot in S^3?*

It should be noted that if a knot K in S^3 admits a longitudinal, Seifert fibered surgery, then K is a fibered knot [10, Corollary 8.19] (see also [27]). If K is a (p,q)-torus knot $T_{p,q}$ or a connected sum of two torus knots $T_{p,q} \sharp T_{p,-q}$, then a longitudinal surgery on K produces a Seifert fiber space. On the contrary, it has been expected by Teragaito [29] that there are no longitudinal, Seifert fibered surgeries on hyperbolic knots (see also [27], [28]).

In Section 2, we give an answer to Question 1.2 by demonstrating:

Theorem 1.3 *There is an infinite family of hyperbolic, fibered knots in S^3 each of which admits a longitudinal Seifert fibered surgery.*

From Proposition 1.1, we see that the monodromies of fibered knots in Theorem 1.3 are isotopic to pseudo-Anosov automorphisms with prong one singularity at the boundary.

Another motivation for studying longitudinal Seifert fibered surgeries comes from a relationship between boundary slopes and Seifert fibered surgery slopes, see Proposition 3.1.

A slope γ on $\partial E(K)$ is called a *boundary slope* if a representative of γ is a boundary component of an essential surface in the exterior $E(K)$. Let K be a *small* knot (i.e., its exterior contains no closed essential surface) and γ a boundary slope for K. Then it follows from [7, Theorem 2.0.3] that γ cannot be a cyclic surgery slope, in particular, the result of γ-surgery on K, denoted by $(K;\gamma)$, is not a lens space. *Can* $(K;\gamma)$ *be a small Seifert* *fiber space* (*i.e., a Seifert fiber space over S^2 with three exceptional fibers*)*?* Applying [7, Theorem 2.0.3] again, we see that if such a situation occurs, then the boundary slope γ must be a longitudinal slope (Proposition 3.1).

For this remaining possibility, since those knots given in Theorem 1.3 turns out to be small by Lemma 3.2, we have:

Corollary 1.4 *There exists a small hyperbolic knot in S^3 such that $(K;\gamma)$ is a small Seifert fiber space for some boundary slope γ.*

Now let us turn to the second problem which appeared when we studied Question 1.2. Recall that if a hyperbolic, fibered knot K in S^3 admits a longitudinal Seifert fibered surgery, then the dual knot (i.e., the core of the filled solid torus) is a section in a Seifert fibered, surface bundle with hyperbolic complement. At the beginning of our study, toward finding a longitudinal Seifert fibered surgery on a hyperbolic knot, we tried to find a section in a Seifert fibered, surface bundle, say $(T_{p,q};0)$, so that its exterior is hyperbolic and embeddable in S^3, see Subsection 2.1. It is interesting to compare this with Osoinach's examples [23] of longitudinal toroidal surgeries from such a viewpoint. He starts with a longitudinal surgery on a connected sum of two figure eight knots $4_1\sharp 4_1$. His construction shows that there exist infinitely many sections in $(4_1\sharp 4_1;0)$ each of whose complement is hyperbolic and embeddable in S^3.

For the hyperbolicity condition on sections, in more general setting, we address:

Question 1.5 *Can we describe the positions of hyperbolic sections in a Seifert fibered, surface bundle over the circle?*

To make precise, we consider the following setting.

Let F be a closed, orientable surface of genus $g \geq 2$, and let f be an automorphism of F with $f(x_0) = x_0$ for some point $x_0 \in F$. (For later convenience, we assume that the condition "$f(x_0) = x_0$" implies also that "$f(D_0) = D_0$" for some small diskal neighborhood D_0 of x_0.) Let us consider $F \times [0,1]$ and a monotone arc t connecting $(x_0,0)$ and $(x_0,1)$; t is oriented from $(x_0,0)$ to $(x_0,1)$. Then by taking a mapping torus $M_f = (F \times [0,1])/\{(x,0) = (f(x),1)\}$, which is a surface bundle over the circle, we obtain a section s from t by identifying its endpoints. Projecting $t \subset F \times [0,1]$ into F, we obtain a closed oriented curve c based at x_0, which represents an element in $\pi_1(F,x_0)$. We call c a *projection* of the section s.

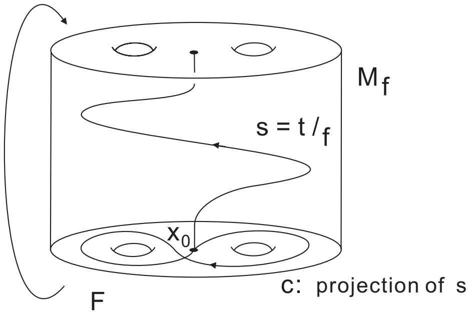

Figure 2: Section and its projection

Conversely for a given closed, oriented curve c, we have a section s whose projection is c, unique up to level preserving isotopy. Henceforth we denote such a section by s_c. Furthermore, if $[c_1] = [c_2] \in \pi_1(F, x_0)$, then s_{c_1} and s_{c_2} are (level preservingly) isotopic in M_f. It should be noted that for $\bar{c}$ (the curve c with the opposite orientation) we have a section $s_{\bar{c}}$ which is not isotopic to s_c generically. Note that M_f is a small Seifert fiber space if and only if the monodromy f is irreducible and periodic ([16, Lemma VI.31], [13]).

The next result gives necessary and sufficient conditions for a section in a small Seifert fibered, surface bundle M_f over the circle being hyperbolic, which answers Question 1.5 in the case where M_f is a small Seifert fiber space.

Theorem 1.6 *Let F be a closed, orientable surface of genus ≥ 2 and f an irreducible, periodic automorphism of period p with $f(x_0) = x_0$ for some point $x_0 \in F$. Let s_c be a section in M_f containing $(x_0, 0) = (x_0, 1)$ whose projection is c. Then the following three conditions are equivalent.*

(1) *s_c is hyperbolic.*

(2) $[c] f_*([c]) \cdots f_*^{p-1}([c]) \neq 1 \in \pi_1(F, x_0)$.

(3) $[c] \neq [\bar{\gamma} * (f \circ \gamma)]$ *in $\pi_1(F, x_0)$ for any path γ from x_i to x_0, where x_i is a fixed point of f.*

Remark. If $\mathrm{Fix}(f) = \{x_0\}$, then $x_i = x_0$ and γ is a closed path based at x_0 and the last condition is simplified to the condition "$[c] \neq \alpha^{-1} f_*(\alpha)$ for any $\alpha \in \pi_1(F, x_0)$". Here f_* denotes the induced isomorphism of $\pi_1(F, x_0)$.

Then which curve c on F based at x_0 satisfies the condition (2) (or equivalently (3))? We say that an element $[c] \in \pi_1(f, x_0)$ is *non-returnable* (with respect to f) if it satisfies the condition (2) (or equivalently (3)) and we would like to pose:

Question 1.7 *Assume that* $[c] \neq 1 \in \pi_1(F, x_0)$. *Then is* $[c]$ *or* $[c]^{-1}$ *non-returnable?*

By introducing a geometric condition, we will give a partial answer to this.

Define a *length* function L of $\pi_1(F, x_0)$ as follows. First we choose an $\langle f \rangle$-invariant hyperbolic metric on F; for existence of such a metric, see [26, Section 2]. Then each $\alpha \in \pi_1(F, x_0)$ can be represented by a unique *geodesic closed path* γ_α i.e., a curve $\gamma_\alpha : I \to F$ such that $\gamma_\alpha(0) = \gamma_\alpha(1) = x_0$ and the restriction $\gamma_\alpha|_{(0,1)}$ on $(0, 1)$ is a geodesic. Then we put $L(\alpha) = \text{length}(\gamma_\alpha)$. Note that $L(\alpha^{-1}) = L(\alpha)$.

Theorem 1.8 *Let* F *be a closed, orientable surface of genus* ≥ 2 *and* f *a periodic automorphism of period* $p > 2$ *such that* $f(x_0) = x_0$. *Then there is a constant* C_p *depending on* p *so that if* $L([c]) > C_p$, *then* $[c]$ *or* $[c]^{-1}$ *is non-returnable.*

Regard the hyperbolic plane $\mathbb{H}^2$ as the universal cover of F. Then the preimage of $x_0 (\in F)$ in $\mathbb{H}^2$ belonging to a disk of radius C_p consists of only finitely many points. This answers Question 1.7 affirmatively with only finitely many exceptions.

Combining Theorem 1.8 with Theorem 1.6, we have:

Corollary 1.9 *Let* F, f *and* C_p *be as in Theorem 1.6. Then if* $L([c]) > C_p$, *then the section* s_c *or* $s_{\bar{c}}$ *is hyperbolic in* M_f.

More precisely, considering the angle from $\dot{\gamma_{[c]}}(0)$ to $\dot{\gamma_{[c]}}(1)$, we can detect s_c is hyperbolic or $s_{\bar{c}}$ is hyperbolic, for details see the proof of Theorem 1.8.

By a numerical computation, we have the following table of approximations of the constants C_p $(3 \leq p \leq 15)$.

p	3	4	5	6	7	8	9	10	11	12	13	14	15
C_p	2.63	3.23	3.69	4.06	4.37	4.64	4.87	5.08	5.28	5.45	5.61	5.76	5.90

Table 1: The constants C_p $(3 \leq p \leq 15)$

In Theorem 1.6, we assume that the monodromy map is irreducible. We can prove also the following result for reducible, periodic monodromies.

An element $\alpha \in \pi_1(F, x_0)$ is said to be *filling* if any representative of α intersects every essential simple closed curve in F.

Theorem 1.10 *Let F be a closed, orientable surface of genus ≥ 2 and f a reducible, periodic automorphism of period p with $f(x_0) = x_0$ for some point $x_0 \in F$. Let s_c be a section in M_f containing $(x_0, 0) = (x_0, 1)$ whose projection is c. Then the following two conditions are equivalent.*

(1) *s_c is hyperbolic.*

(2) *$[c]f_*([c]) \cdots f_*^{p-1}([c]) \in \pi_1(F, x_0)$ is filling.*

In Section 6, we will apply Theorems 1.6, 1.10 and Corollary 1.9 to surface-automorphisms.

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2 Longitudinal Seifert fibered surgeries on knots

Let K be a knot in S^3. Assume that $(K; 0)$ is a Seifert fiber space. Then $(K; 0)$ contains an essential, non-separating surface, and hence $(K; 0)$ is a surface bundle over the circle. From [10, Corollary 8.19], we see that K is a fibered knot in S^3. We shall say that a Seifert fiber space is of *type* $S^2(n_1, n_2, n_3)$ if it has a Seifert fibration over S^2 with three exceptional fibers of indices n_1, n_2 and n_3 ($n_i \geq 2$).

2.1 Torus knots

Let $T_{p,q}$ be a (p, q)-torus knot in S^3 such that $q > p \geq 2$ and $(p, q) \neq (2, 3)$. It is known that $T_{p,q}$ is a fibered knot with fiber genus $\frac{(p-1)(q-1)}{2}$ and a periodic monodromy of period pq. Then the result of a longitudinal surgery $(T_{p,q}; 0)$ is regarded as a mapping torus $M_f(p, q) = (F \times [0, 1])/\{(x, 0) = (f(x), 1)\}$, where F is the capped off closed surface of genus $\frac{(p-1)(q-1)}{2}$ and f is the capped off monodromy of period pq. It is easy to see that M_f is a Seifert fiber space of type $S^2(p,q,pq)$. In $M_f(p,q)$ the dual knot $T^*_{p,q}$ (i.e., the core of the filled solid torus) is obtained from a vertical segment $t_0 = \{x_0\} \times [0,1]$ by identifying their endpoints for some point $x_0 \in F$. Note that f fixes $x_0 \in F$. Then $T^*_{p,q}$ gives a section in M_f whose projection represents an element in $\pi_1(F, x_0)$ of length zero.

As we mentioned in Section 1, we are interested in finding a hyperbolic section whose exterior is embeddable in S^3. Now we explain how we apply Corollary 1.9 to find hyperbolic sections, say in $M_f(2,5) = (T_{2,5}; 0)$. Let us choose a curve c on the fiber surface as in Figure 3 so that $L([c]) > C_{10} \doteq 5.1$ according to the table in Section 1. (We can verify the curve c in Figure 3 satisfies $L([c]) > C_{10} \doteq 5.1$, but we omit the details here.) Then a section s_c or $s_{\bar{c}}$ is hyperbolic in $(T_{2,5}; 0)$ by Corollary 1.9.

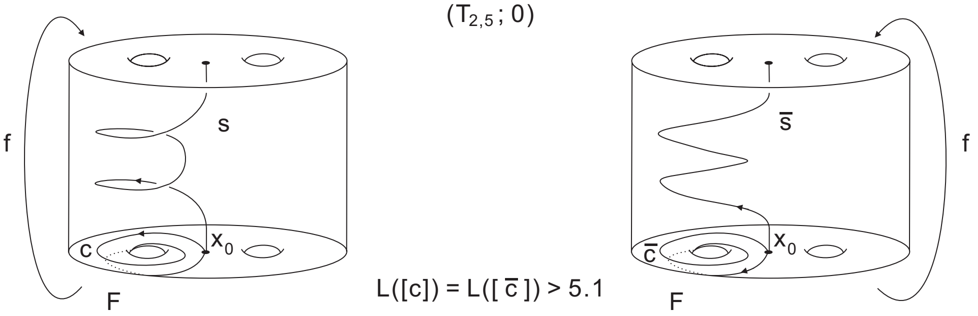


Figure 3: s or $\bar{s}$ is a hyperbolic section in $(T_{2,5}; 0)$

Since with only finitely many exceptions $L([c]) > C_{10}$, we have infinitely many hyperbolic sections in $(T_{2,5}; 0)$. Similarly we have infinitely many hyperbolic sections in $(T_{p,q}; 0)$, but the following question is still open.

Question 2.1 *Can we find a hyperbolic section $s \subset M_f(p,q)$ so that its complement can be embedded in S^3?*

It should be noted that if $(p,q) = (2,3)$, then $(M_f(2,3) = (T_{2,3}; 0)$ is a torus bundle over the circle and there is no hyperbolic section in $M_f(2,3) = (T_{2,3}; 0)$, see Addendum 6.4.

2.2 Satellite knots

Proposition 2.2 *Let K be a satellite, fibered knot in S^3 (i.e., a fibered knot whose exterior contains an essential torus). Then no longitudinal surgery on K yields a small Seifert fiber space.*

This result follows from the argument in [20, Theorem 1.4], but we give an alternative proof here. We begin by showing the result below, which will be used also in Section 4.

Let F be a closed, orientable surface with negative Euler characteristic and $M_f = (F \times [0,1])/\{(x,0) = (f(x),1)\}$ a mapping torus of F with a periodic gluing map f which leaves a point x_0 and its diskal neighborhood D_0 invariant.

Lemma 2.3 *If M_f is a small Seifert fiber space, then for any section s in M_f, $M_f - \mathrm{int}N(s)$ is atoroidal, i.e., there is no essential torus in $M_f - \mathrm{int}N(s)$.*

Proof. We denote a punctured surface $F - \mathrm{int}D_0$ by $\hat{F}$. We identify F (resp. $\hat{F}$) with $F \times \{0\} = F \times \{1\} \subset M_f$ (resp. $F \times \{0\} - \mathrm{int}D_0 \times \{0\} = F \times \{1\} - \mathrm{int}D_0 \times \{1\} \subset M_f - \mathrm{int}N(s)$).

Suppose for a contradiction that $M_f - \mathrm{int}N(s)$ contains an essential torus T for some section s. Isotope T so that it intersects $\hat{F}$ transversely and that if we cut M_f along $\hat{F}$, T is cut into some annuli $A_1, \dots, A_m$ such that each A_i runs from $F \times \{0\}$ to $F \times \{1\}$.

If T is incompressible in M_f, then since M_f is a small Seifert fiber space, it must be non-separating ([16, Example VI.13]) and isotopic to a horizontal torus. This implies that the Euler characteristic of the base orbifold of the Seifert fiber space M_f is zero. Thus M_f has $\mathbb{E}^3$-geometry. On the other hand, since F has negative Euler characteristic, it has $H^2 \times \mathbb{R}$-geometry, see [26, Sect 4]. This contradicts the uniqueness of the geometric structure on M_f ([26, Theorem 5.2]).

Thus T is compressible in M_f. Then $\partial A_i \cap (F \times \{0\})$ bounds a disk Δ_0 in F intersecting s exactly once for some $i \in \{1, \dots, m\}$, and hence $\partial A_i \cap (F \times \{1\})$ also bounds a disk Δ_1 in F intersecting s exactly once. The irreducibility of $F \times [0,1]$ shows that $\Delta_0 \cup A_i \cup \Delta_1$ bounds a 3-ball B such that the pair (B, t) is a trivial tangle, where t is an arc obtained from s by cutting at $(x_0, 0) = (x_0, 1)$. This then implies that $m = 1$ and T is boundary-parallel in $M_f - \mathrm{int}N(s)$. This contradicts the choice of T. □

Proof of Proposition 2.2. Since K is a satellite knot, in particular, it is neither a trefoil knot nor a figure-eight knot, its fiber genus ≥ 2 ([5, Proposition 5.14]). Assume for a contradiction that $M_f = (K;0)$ is a small Seifert fiber space. Then by Lemma 2.3 the exterior of the section $s = K^*$ (the core of the filled solid torus) is atoroidal. Hence $S^3 - \mathrm{int}N(K) = (K;0) - \mathrm{int}N(K^*) = M_f - \mathrm{int}N(s)$ is atoroidal, a contradiction. $\square$

Note that a non-small Seifert fiber space can be obtained from S^3 by a longitudinal surgery on fibered, satellite knot. For instance, 0-surgery on the connected sum of torus knot $T_{p,q}\sharp T_{-p,q}$ $(q > p \geq 2)$ yields a Seifert fiber space over S^2 with four exceptional fibers of indices p, q, p and q.

2.3 Hyperbolic knots

In this subsection, we prove Theorem 1.3. The proof follows from Lemmas 2.4–2.7 below.

Let $k \cup t_1 \cup t_2 \cup t_3$ be the four component link of Figure 4; each component is a trivial knot in S^3.

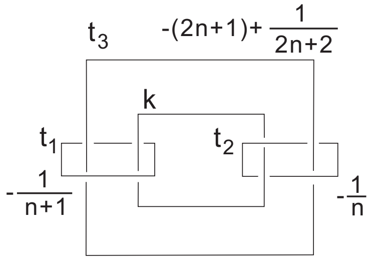


Figure 4: Surgery description of K_n

Let K_n be a knot obtained from k by performing $-\frac{1}{n+1}$, $-\frac{1}{n}$ and $-(2n+1) + \frac{1}{2n+2}$-surgeries on t_1, t_2 and t_3, respectively.

Lemma 2.4 *The knot K_n is lying in S^3 and $(K_n;0)$ is a manifold obtained from S^3 by performing 1, $-\frac{1}{n+1}$, $-\frac{1}{n}$ and $-(2n+1) + \frac{1}{2n+2}$-surgery on k, t_1, t_2 and t_3, respectively.*

Proof. After $-\frac{1}{n+1}$-surgery on t_1 (i.e., $n+1$-twist along t_1) and $-\frac{1}{n}$-surgery on t_2 (i.e., n-twist along t_2), $k \cup t_3$ become a 2-bridge link in (new) S^3 with their linking number $(n+1) - n = 1$ and new framings $2n+2$ and $\frac{1}{2n+2}$ on k and t_3, respectively (Figure 5).

Then after $-(2n+2)$-twist along t_3, we obtain a knot K_n in S^3 with the framing $(2n+2) - 1^2 \times (2n+2) = 0$. $\square$

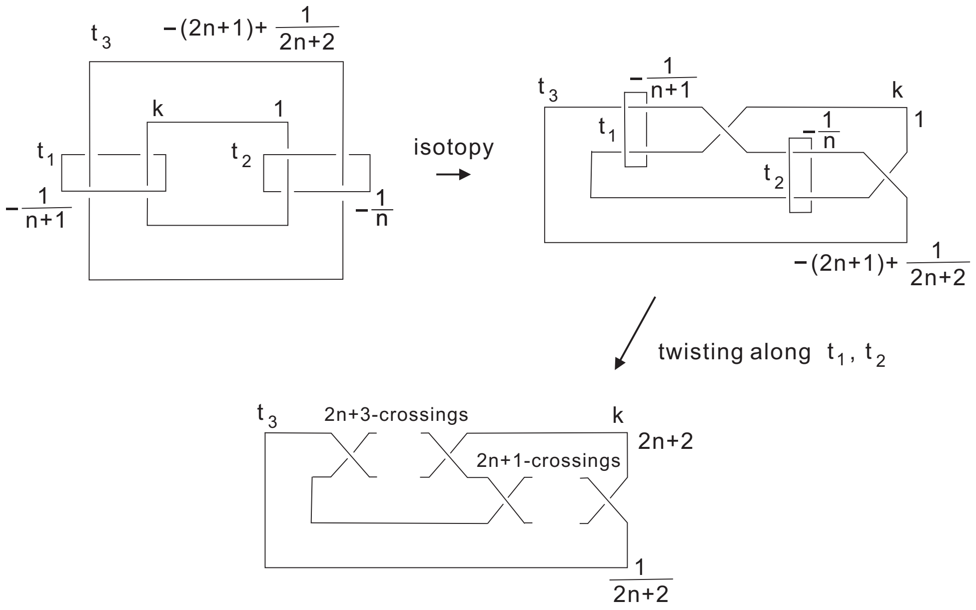


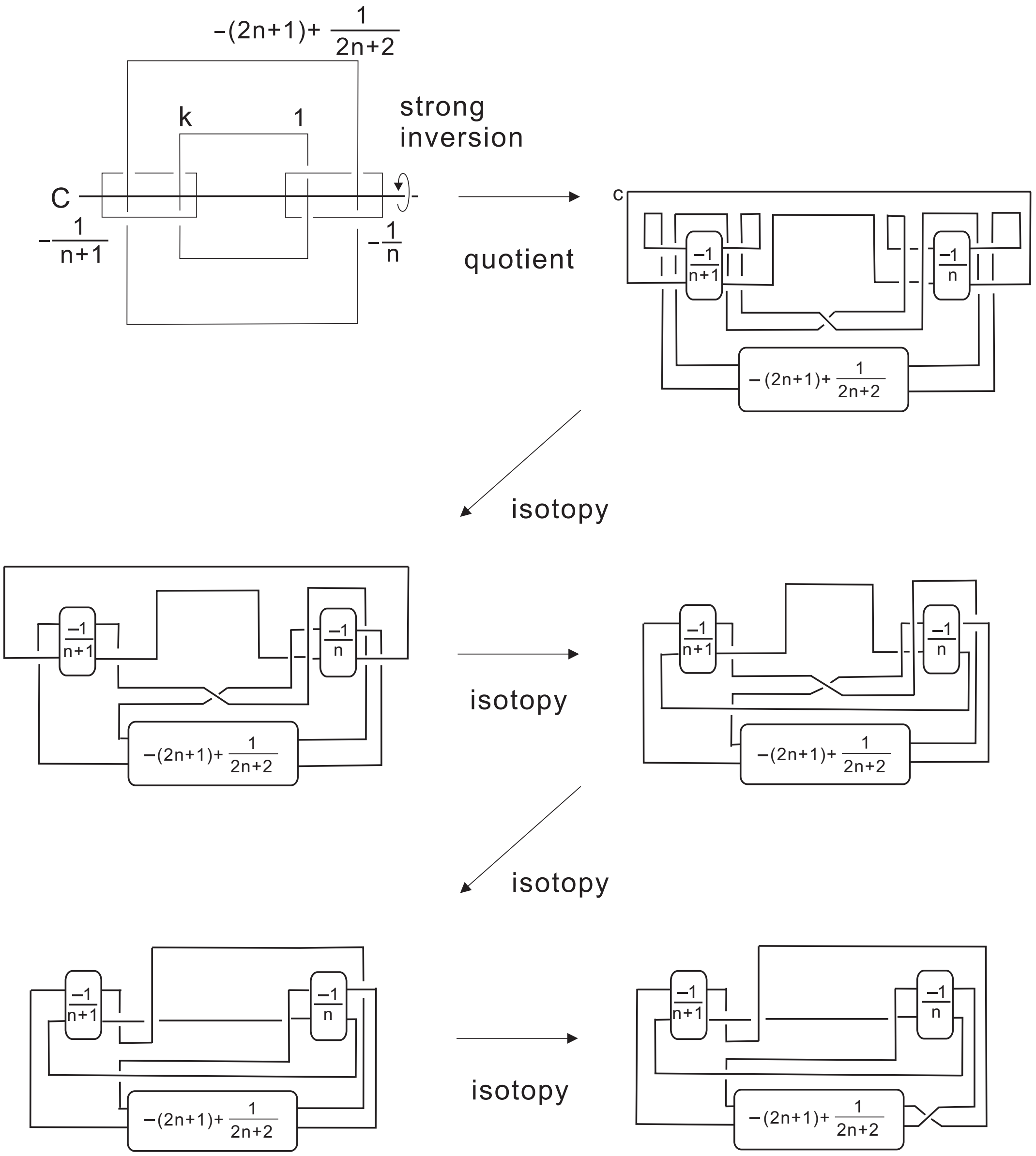
Figure 5: Modification of the surgery description

Lemma 2.5 *The resulting* 3*-manifold* $(K_n; 0)$ $(n \neq 0, -1, -2)$ *is a small Seifert fiber space of type* $S^2(|2n+1|, |2n+3|, |(2n+1)(2n+3)|)$.

Proof. To prove this we take the quotient by the strong inversion of S^3 as shown in Figure 6. Then we obtain a branch knot c' which is the image of the axis C. The Montesinos trick ([21], [2]) shows that r_i-surgery on t_i in the upstairs corresponds to r_i-untangle surgery in the downstairs, i.e., a replacement of 1/0-untangle by r_i-untangle. (We adopt Bleiler's convention [3] on the parametrization of rational tangles.) These untangle surgeries convert the knot c' into a link c (Figure 6).

We follow the sequence of isotopies in Figures 6 and 7. Finally we obtain a Montesinos link $M(\frac{n+1}{2n+3}, -\frac{n+1}{2n+1}, \frac{2n+2}{(2n+1)(2n+3)})$. Since $(K_n; 0)$ is a branched cover of S^3 branched over the Montesinos $M(\frac{n+1}{2n+3}, -\frac{n+1}{2n+1}, \frac{2n+2}{(2n+1)(2n+3)})$, $(K_n; 0)$ is a Seifert fiber space of type $S^2(|2n+1|, |2n+3|, |(2n+1)(2n+3)|)$. □

Since $(K_n; 0)$ is a Seifert fiber space, as we mentioned at the beginning of this section, it follows from [10, Corollary 8.19] that K is a fibered knot in S^3.



$-(2n+1)+\frac{1}{2n+2}$
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Figure 6:

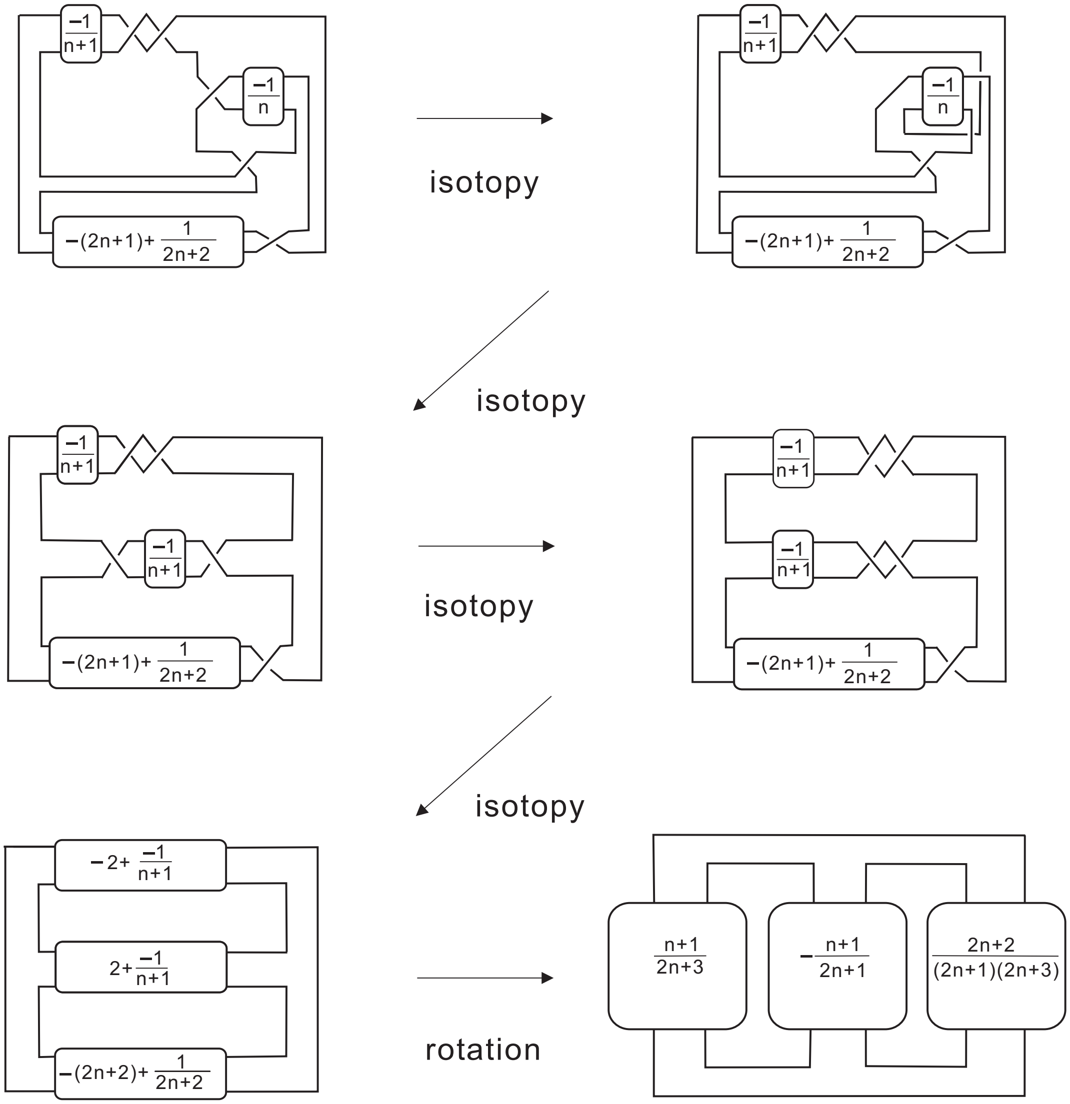

Figure 7: Continued from Figure 6

Lemma 2.6 *The Seifert fiber space* $(K_n; 0)$ $(n \neq 0, -1, -2)$ *cannot be obtained from* S^3 *by Dehn surgery on any torus knot.*

Proof. Suppose that $(K_n; 0)$ is also obtained from S^3 by a longitudinal surgery on a torus knot $T_{p,q}$ $(2 \leq |p| < q)$. Then $(K_n; 0) = (T_{p,q}; 0) = S^2(|p|, q, |pq|)$, and hence, referring Lemma 2.5 we have $\{|p|, q\} = \{|2n+1|, |2n+3|\}$. Then the exceptional fiber of the maximal index $|(2n+1)(2n+3)|$ in $(K_n; 0)$ is isotopic to the dual knot $T^*_{p,q}$, hence the exterior of the exceptional fiber in $(K_n; 0)$, denoted by E, is the exterior of $T_{|2n+1|,|2n+3|}$ in S^3.

Recall from the proof of Lemma 2.5 that E is a Seifert fiber space over the disk with Seifert invariant $\frac{n+1}{2n+3}, -\frac{n+1}{2n+1} \in \mathbb{Q}/\mathbb{Z}$. Since E is a torus knot space, we see that

$$(2n+3)\{-(n+1) + \ell(2n+1)\} + (2n+1)\{(n+1) + k(2n+3)\} = \pm 1$$

for some integers k, ℓ, see [4, Proposition 4.2 (d)]. Hence we have

$$(2n+1)(2n+3)(k+\ell) - 2(n+1) = \pm 1$$

for some integers k, ℓ.

If $k + \ell \neq 0$, then since $|(2n+1)(2n+3)(k+\ell)| \geq |(2n+1)(2n+3)| > 2|n+1| + 1$, $(2n+1)(2n+3)(k+\ell) - 2(n+1) \neq \pm 1$ for any integers k, ℓ, a contradiction. If $k + \ell = 0$, then we would have $-2(n+1) = \pm 1$, a contradiction.

It follows that $(K_n; 0)$ cannot be obtained from S^3 by Dehn surgery on any torus knot. □

Note that if $n = 0, -1$ or -2, then K_n is a trivial knot and $(K_n; 0) \cong S^2 \times S^1$.

For remaining n's, it follows from Lemma 2.6 that K_n $(n \neq 0, -1, -2)$ is not a torus knot.

Moreover, we have:

Lemma 2.7 K_n *is a hyperbolic knot for* $n \neq 0, -1, -2$.

Proof. To prove that K_n is a hyperbolic knot, following Thurston's uniformization theorem ([22], [30]) and the torus theorem ([17], [18]), it is sufficient to check that K_n is neither a torus knot nor a satellite knot. As we mentioned above, K_n $(n \neq 0, -1, -2)$ is not a torus knot. Moreover, since $(K_n; 0)$ is a small Seifert fiber space (Lemma 2.5), Proposition 2.2 shows that K_n is not a satellite knot. □

This completes a proof of Theorem 1.3. □

It should be mentioned here that Weeks' computer program SnapPea [34] was quite useful at the beginning of our study in Subsection 2.3.

3 Boundary slopes and Seifert fibered slopes

Let K be a small knot and γ a boundary slope of K. Then as a consequence of [7, Theorem 2.0.3], $(K;\gamma)$ cannot be a lens space.

For small Seifert fibered slopes, we have the following.

Proposition 3.1 *Let K be a small knot in S^3. If a surgery on K along a boundary slope γ yields a small Seifert fiber space, then K is a fibered knot and γ is a fiber slope (i.e., a longitudinal slope).*

Proof. Since K is a small knot and $(K;\gamma)$ is irreducible ([16, VI.13.Example]), referring to the possibilities of the conclusion of [7, Theorem 2.0.3], we see that $(K;\gamma)$ is a Haken manifold or $E(K)$ fibers over S^1 with fiber a planar surface having boundary slope γ. In the former case, [16, VI.13.Example] shows that $H_1((K;\gamma))$ is infinite and $(K;\gamma)$ is a surface bundle over S^1 and hence, γ is a longitudinal slope. Then [10, Corollary 8.19] shows that K is a fibered knot in S^3 with fiber slope γ. In the latter case, $(K;\gamma)$ contains a non-separating 2-sphere, in particular it is reducible, a contradiction. □

If each hyperbolic knot in the family given in Theorem 1.3 is small, then we establish Corollary 1.4 which shows that the situation described in Proposition 3.1 can happen.

Lemma 3.2 *Each hyperbolic knot K_n $(n \neq 0, -1, -2)$ is a small knot.*

Proof. As we observed in the proof of Lemma 2.4, the knot K_n is obtained from a component k of a two-bridge link $L([-2n-1, 2n+3]) = k \cup t_3$ by twisting along the other component t_3, see Figure 5.

Note that $L([-2n-1, 2n+3])$ is isotopic to $L([-2n-3, 2n+1])$, and isotopic to the mirror image of $L([2n+1, -2n-3])$ by their symmetries.

Now assume for a contradiction that K_n $(n \neq 0, -1, -2)$ is not small. Then there is a closed essential surface F in $E(K_n)$. Since there is no closed essential surface in a two-bridge link exterior ([14, Theorem 1 and its Remarks (1)]), F intersects the dual

t_3^* of t_3 in $E(K_n)$. We may isotope F so that $F \cap N(t_3^*)$ consists of meridian disks of $N(t_3^*)$ and the number of such disks is minimal. Put $S = F \cap E(K_n \cup t_3^*)$, which is a properly embedded essential surface in $E(K_n \cup t_3^*) = E(k \cup t_3)$. Then a component of ∂S has a slope $\frac{1}{2n+2}(\neq \frac{1}{0})$ on $\partial N(t_3)$. If S is meridionally compressible, i.e., there is a disk $D \subset S^3$ such that $D \cap S = \partial D$ is essential in S and D meets K_n transversely in one point, then apply meridional surgeries until it is meridionally incompressible. It is easy to check that meridional surgeries preserve essentiality of surfaces. Thus we obtain an essential, meridionally incompressible surface S in the 2-bridge link exterior $E(k \cup t_3)$ satisfying property $(*)$:
(i) $\partial S \cap \partial N(k) = \emptyset$ or $\partial S \cap \partial N(k)$ has slope $\frac{1}{0}$, and
(ii) $\partial S \cap \partial N(t_3)$ has slope $\frac{1}{2n+2}$.

Now we apply a classification of such surfaces in a 2-bridge link exterior due to Goda, Hayashi and Song [12] which is based on [9]. For adaptation of their setting, if $2n+1 > 0$ (i.e., $-2n-1 < 0$), then we once take the mirror image $L([2n+1, -2n-3])$ of $L([-2n-1, 2n+3])$ and then to obtain correct boundary slopes, we change the signs of boundary-slopes given in [12]. Then we have the following tables for pairs of boundary slopes on $\partial N(k)$ and $\partial N(t_3)$ for some integers α, β and m; $(\alpha, \beta) \neq (0,0)$. In the table, we use the $(\text{longitude}, \text{meridian})$-coordinates with multiplication; The usual boundary slope can be obtained by dividing each coordinate by their greatest common divisor.

slope on $\partial N(k)$ (resp. $\partial N(t_3)$)	$\partial N(t_3)$ (resp. $\partial N(k)$)
$(\beta, (2n+3)\beta - 2m)$	$(\beta, (2n+1)\beta + 2m)$
(α, β)	$(\beta, -(2n+1)\alpha)$
$(\alpha, (n+1)\alpha - n\beta)$	$(\beta, -n\alpha + (n+1)\beta)$
(α, β)	(β, α)
$(\alpha, (n+1)\alpha + n\beta)$	$(\beta, n\alpha + (n+3)\beta)$
$(\alpha, (2n+1)\alpha)$	$(\beta, (2n+3)\beta)$
$(\alpha, (2n+1)\alpha)$	$(\beta, (2n+1)\alpha + 2\beta)$
$(\alpha, (2n+3)\alpha)$	$(\beta, (2n+1)\beta)$

Table 2: Boundary slopes in case of $2n+1 > 0$

slope on $\partial N(k)$ (resp. $\partial N(t_3)$)	$\partial N(t_3)$ (resp. $\partial N(k)$)
$(\beta, (2n+1)\beta + 2m)$	$(\beta, (2n+3)\beta - 2m)$
(α, β)	$(\beta, -(2n+3)\alpha)$
$(\alpha, (n+2)\alpha - (n+1)\beta)$	$(\beta, -(n+1)\alpha + (n+2)\beta)$
(α, β)	(β, α)
$(\alpha, (n+2)\alpha + (n+1)\beta)$	$(\beta, (n+1)\alpha + n\beta)$
$(\alpha, (2n+3)\alpha)$	$(\beta, (2n+1)\beta)$
$(\alpha, (2n+3)\alpha)$	$(\beta, (2n+3)\alpha - 2\beta)$
$(\alpha, (2n+1)\alpha)$	$(\beta, (2n+3)\beta)$

Table 3: Boundary slopes in case of $2n+1<0$

Referring these tables, simple computation shows that there is no essential surface S with property $(*)$. It follows that the hyperbolic knot K_n $(n \neq 0, -1, -2)$ is a small knot as desired. □

4 Sections and their projections on the fiber surface

4.1 Proof of Theorem 1.6

The goal of this subsection is to prove Theorem 1.6.

Let F, f and s_c be as in Theorem 1.6. Recall that we assume that $f(D_0) = D_0$ for some small diskal neighborhood D_0 of x_0. Denote by $N(s_c)$ a tubular neighborhood of s_c in M_f such that $N(s_c) \cap F \times \{0\} = D_0 \times \{0\}$, $N(s_c) \cap F \times \{1\} = D_0 \times \{1\}$.

Proof of (1) $\Rightarrow$ (2).

Assume for a contradiction that $[c]f_*([c]) \cdots f_*^{p-1}([c]) = 1 \in \pi_1(F, x_0)$.

Let us consider a pair $(F \times [0,1], t_c)$, where t_c is a monotone arc in $F \times [0,1]$ connecting $(x_0, 0)$ and $(x_0, 1)$, which gives the section s_c in M_f. Take p copies of the pair $(F \times [0,1], t_c)$ and construct a manifold $\widetilde{M_f}$ as follows. Let us denote the i-th copy of $(F \times [0,1], t_c)$ by $(F^i \times [0,1], t_c^i)$, and identify $(f(x), 1) \in F^i \times \{1\}$ and $(x, 0) \in F^{i+1} \times \{0\}$, where $i = 1, \ldots, p$ (mod p). See Figure 8.

Then $\widetilde{M_f}$ is a p-fold cyclic covering of M_f with a covering projection $q : \widetilde{M_f} \to M_f$. Note that $\widetilde{M_f}$ is a surface bundle over the circle with the trivial monodromy (i.e., a trivial surface bundle) and the preimage $\tilde{s_c}$ of s_c is a section in $\widetilde{M_f}$ such that the associated monotone arc $\tilde{t_c}$ has the form $t_c \cup (f \times id.)(t_c) \cup \cdots \cup (f^{p-1} \times id.)(t_c)$, see Figure 8. Hence projecting $\tilde{t_c}$ into F, we obtain a closed curve $c * f(c) * \cdots * f^{p-1}(c)$ based at x_0.

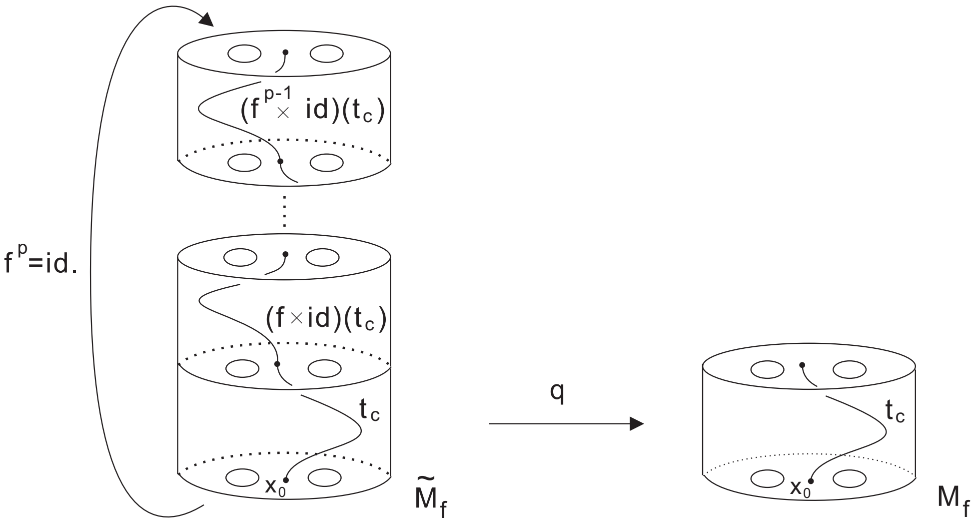

Figure 8: covering construction

Since $M_f - N(s_c)$ is hyperbolic, we can pull back the hyperbolic metric of $M_f - N(s_c)$ to that of $\widetilde{M_f} - N(\tilde{s_c})$ so that $\widetilde{M_f} - N(\tilde{s_c})$ is hyperbolic.

On the other hand, since $[c]f_*([c]) \cdots f_*^{p-1}([c]) = 1 \in \pi_1(F, x_0)$, the monotone arc $\tilde{t_c}$ can be isotoped to a vertical segment, and hence $\widetilde{M_f} - \mathrm{int}N(\tilde{s_c})$ is a product S^1-bundle over $F - \mathrm{int}D_0$. In particular, it is Seifert fibered, contradicting the hyperbolicity of $\widetilde{M_f} - N(\tilde{s_c})$. $\square$

Proof of (2) $\Rightarrow$ (3).

Suppose for a contradiction that we have $[c] = [\bar{\gamma} * (f \circ \gamma)] \in \pi_1(F, x_0)$ for some path γ from x_i to x_0. Then $f_*([c]) = [(f \circ \bar{\gamma}) * (f^2 \circ \gamma)]$, $f_*^2([c]) = [(f^2 \circ \bar{\gamma}) * (f^3 \circ \gamma)]$, $\dots$, $f_*^{p-1}([c]) = [(f^{p-1} \circ \bar{\gamma}) * (f^p \circ \gamma)] = [(f^{p-1} \circ \bar{\gamma}) * \gamma]$. Since $(f^m \circ \gamma) * (f^m \circ \bar{\gamma}) \simeq f^m \circ (\gamma * \bar{\gamma})$ is homotopic to the constant path e_{x_i} based at x_i, we have $[c]f_*([c]) \cdots f_*^{p-1}([c]) = [e_{x_0}] = 1$, contradicting the assumption. $\square$

Proof of (3) $\Rightarrow$ (1).

To prove that the section s_c is hyperbolic, it is sufficient to show that $M_f - \mathrm{int}N(s_c)$ is atoroidal and not Seifert fibered by Thurston's uniformization theorem ([22], [30]) and the torus theorem ([17], [18]).

Since M_f is a small Seifert fiber space, from Lemma 2.3, we have:

Lemma 4.1 *For any section s_c, $M_f - \mathrm{int}N(s_c)$ is atoroidal.*

Next we show that $M_f - \mathrm{int}N(s_c)$ is not Seifert fibered for any section s_c such that c satisfies the condition (3) in Theorem 1.6. Suppose for a contradiction that we have a section s_c such that c satisfies the condition (3) in Theorem 1.6, but its exterior $M_f - \mathrm{int}N(s_c)$ is Seifert fibered.

A general fact below shows that we can extend the Seifert fibration of $M_f - \mathrm{int}N(s_c)$ to that of M_f so that s_c is a Seifert fiber in the extended Seifert fibration of M_f.

Lemma 4.2 *Let M be an irreducible 3-manifold and k a knot in M. If $M - \mathrm{int}N(k)$ is Seifert fibered, then M admits a Seifert fibration in which k is a Seifert fiber.*

Proof. If the meridian of k is not a fiber in the Seifert fibration of $M - \mathrm{int}N(k)$, then we can extend it to a Seifert fibration of M so that k is a Seifert fiber in the (extended) Seifert fibration. Thus in the following, we assume that the meridian of k is a fiber.

Let B be the base orbifold of $M - \mathrm{int}N(k)$. If B is a disk or a disk with one singular point, then $M - \mathrm{int}N(k)$ is homeomorphic to $S^1 \times D^2$. Then by choosing a suitable Seifert fibration of $M - \mathrm{int}N(k)$ so that the meridian of k is not a regular fiber, we can reduce to the first situation. Otherwise, there exits an essential arc properly embedded in the orbifold B, i.e., a properly embedded arc which does not cut off a disk without singular points. This implies that M is reducible. This contradicts the assumption. $\square$

Then we have the following, which contradicts the assumption of (3) in Theorem 1.6.

Lemma 4.3 *The projection c of s_c satisfies $[c] = [\bar{\gamma} * (f \circ \gamma)]$ in $\pi_1(F, x_0)$ for some path γ from x_i to x_0.*

Proof. Recall that s_c is a Seifert fiber in the (extended) Seifert fibration of M_f.

On the other hand, we have the natural Seifert fibration of M_f for the periodic automorphism f in which each fiber comes from vertical segments in $F \times [0, 1]$. Note that since $\mathrm{Fix}(f) = \{x_0, \dots, x_n\}$, only $s_i = (\{x_i\} \times I)/\{(x_i, 0) = (x_i, 1)\}$ $(i = 0, \dots, n)$ are Seifert fibers in M_f which intersect $F \times \{0\}$ algebraically once.

Since F has genus greater than one, the above two Seifert fibrations are isotopic, see [26, Theorem 3.9 and second paragraph in page 441]. In particular, since s_c intersects $F \times \{0\}$ algebraically once, s_c is isotoped to a Seifert fiber s_i for some i.

Hence we have an isotopy $\varphi : [0,1] \times S^1 \to M_f$ such that $\varphi(\{0\} \times S^1) = s_i$ and $\varphi(\{1\} \times S^1) = s_c$. By an isotopy keeping $\{0,1\} \times S^1$ invariant, we may assume that φ is transverse to $F \times \{0\}$. Then since $\varphi(\{0\} \times S^1) = s_i$ and $\varphi(\{1\} \times S^1) = s$ intersect $F \times \{0\}$ exactly once respectively, $\varphi^{-1}(F \times \{0\})$ consists of a single essential arc γ' in $[0,1] \times S^1$ and some circles. By an incompressibility of $F \times \{0\}(= F \times \{1\})$ in M_f, we can isotope φ further so that the circle components of $\varphi^{-1}(F \times \{0\}) \subset I \times S^1$ disappear; hence $\varphi^{-1}(F \times \{0\})$ consists of the single arc γ' in $I \times S^1$.

Cutting $[0,1] \times S^1$ along γ', we obtain a rectangle homeomorphic to $[0,1] \times [0,1]$. Let $\hat{\varphi} : [0,1] \times [0,1] \to F \times [0,1]$ be the map naturally induced from $\varphi : [0,1] \times S^1 \to M_f$, which satisfies that $\hat{\varphi}(\{0\} \times [0,1]) = \{x_i\} \times [0,1]$, $\hat{\varphi}(\{1\} \times [0,1]) = t_c$ and $\hat{\varphi}([0,1] \times \{0\}) = \varphi(\gamma') \subset F \times \{0\}$; here we use the symbol $\varphi(\gamma') \subset F \times \{0\} \subset F \times [0,1]$ to denote a copy of $\varphi(\gamma') \subset M_f$. Note that $\hat{\varphi}([0,1] \times \{1\}) = f(\varphi(\gamma')) \subset F \times \{1\}$. The first and the second equalities implies that $\pi \circ \hat{\varphi}|_{\{0\}\times[0,1]} = e_{x_i} : [0,1] \to F$, where $\pi : F \times [0,1] \to F$ is the natural projection, and $\pi \circ \hat{\varphi}|_{\{1\}\times[0,1]} = c : [0,1] \to F$. Moreover, putting $\gamma = \pi \circ \hat{\varphi}|_{[0,1]\times\{0\}} : [0,1] \to F$, we have $\pi \circ \hat{\varphi}|_{[0,1]\times\{1\}} = f \circ \gamma : [0,1] \to F$. Thus $\pi \circ \hat{\varphi}$ gives the equality $[c] = [\bar{\gamma} * e_{x_i} * (f \circ \gamma)] = [\bar{\gamma} * (f \circ \gamma)] \in \pi_1(F, x_0)$. □

Now we finish a proof of Theorem 1.6. □

4.2 Proof of Theorem 1.10

Let us take a p-fold cyclic covering $\widetilde{M_f}$ of M_f with the covering projection $q : \widetilde{M_f} \to M_f$ as in the proof of Theorem 1.6 so that $\widetilde{M_f}$ is a trivial surface bundle.

Recall that the preimage $\tilde{s_c}$ of s_c is a section in $\widetilde{M_f}$ with a projection $c * f(c) * \cdots * f^{p-1}(c)$ based at x_0.

Proof of (1) ⇒ (2).

Since $M_f - \mathrm{int}N(s_c)$ is hyperbolic, as in the proof of Theorem 1.6, we can pull back the hyperbolic metric of $M_f - N(s_c)$ to that of $\widetilde{M_f} - N(\tilde{s_c})$ so that $\widetilde{M_f} - N(\tilde{s_c})$ is also hyperbolic. Assume for a contradiction that $[c]f_*([c]) \cdots f_*^{p-1}([c]) = [c * f(c) * \cdots * f^{p-1}(c)] \in \pi_1(F, x_0)$ is not filling. Then, by definition, we can find an essential embedded loop ℓ in F which is disjoint from some representative c' of $[c]f_*([c]) \cdots f_*^{p-1}([c])$. Since a section $s_{c'}$ with projection c' is level preservingly isotopic to $\tilde{s_c}$ in $\widetilde{M_f}$, $s_{c'}$ is also hyperbolic in $\widetilde{M_f}$. Let T' be a torus in $\widetilde{M_f}$ obtained from an annulus $\ell \times [0,1]$ by identifying $\ell \times \{0\}$ and $\ell \times \{1\}$, which is incompressible in M_f, because that c' is essential in F. Since T' is disjoint from

$s_{c'}$, T' is lying also in $M_f - \mathrm{int}N(s_{c'})$, and T' is essential in $M_f - \mathrm{int}N(s_c)$. This contradicts the hyperbolicity of $s_{c'}$.

Remark. The argument in the above proof works well in the case where f is irreducible, but in such a case, the condition "$[c]f_*([c])\cdots f_*^{p-1}([c])$ is filling" is equivalent to the condition "$[c]f_*([c])\cdots f_*^{p-1}([c]) \neq 1$".

Proof of (2) $\Rightarrow$ (1).

Assume for a contradiction that the section s_c is not hyperbolic. Since an filling element is clearly nontrivial in $\pi_1(F; x_0)$, the proof of Theorem 1.6 shows that $M_f - \mathrm{int}N(s_c)$ is not Seifert fibered. Then it follows from Thurston's uniformization theorem ([22], [30]) and the torus theorem ([17], [18]), $M_f - \mathrm{int}N(s_c)$ would contain an essential torus T.

Lemma 4.4 *T is also incompressible in M_f.*

Proof. Assume for a contradiction that T is compressible in M_f. Then the argument in the fourth paragraph in the proof of Lemma 2.3 shows that T is boundary-parallel in $M_f - \mathrm{int}N(s_c)$, a contradiction. □

Let $\tilde{T} \subset \widetilde{M_f}$ be a component of $q^{-1}(T)$. Then $q|_{\tilde{T}} : \tilde{T} \to T$ is a covering, hence $\tilde{T}$ is also a torus and essential loop in $\tilde{T}$ is projected to an essential loop in T. Thus we have:

Lemma 4.5 *$\tilde{T}$ is an incompressible torus in $\widetilde{M_f}$, which is disjoint from the section $\tilde{s_c}$.*

We can isotope $\tilde{T}$ so that it intersects $F^1 \times \{0\} (= F^p \times \{1\}) \subset \widetilde{M_f}$ transversely and the number of their components is minimal. For simplicity, henceforth we denote $F^1 \times \{0\} (\subset \widetilde{M_f})$ by F. Then after cutting $\widetilde{M_f}$ along F, we obtain $F \times [0,1]$, $\tilde{t_c}$ and properly embedded essential annuli $A_1, \ldots, A_n$ in $F \times [0,1]$. Further by an isotopy leaving $F \times \{0,1\}$ invariant, we can assume that each A_i is a monotone (meaning no local maxima and minima) annulus. It follows from [15, Lemma 2.1] that $n = 1$ and the annulus $A = A_1$ can be isotoped to a vertical annulus $a_1 \times [0,1]$ by a level preserving isotopy which is the identity on $F \times \{0,1\}$. Under the isotopies, $\tilde{t_c}$ is also isotoped to an arc t' in $F \times [0,1]$ which is disjoint from the vertical annulus $a_1 \times [0,1]$. Note that $p(t') \cap a_1 = \emptyset$. Since $[c]f_*([c])\cdots f_*^{p-1}([c]) = [p(t')] \in \pi_1(F, x_0)$, $[c]f_*([c])\cdots f_*^{p-1}([c])$ would not be filling, a contradiction. □

5 Sections with long projections

Throughout this section, let F be a closed, orientable surface of genus ≥ 2 and $f : F \to F$ an orientation preserving, periodic (not necessarily irreducible) automorphism of period p with $f(x_0) = x_0$ for some point $x_0 \in F$.

Recall that an element $[c]$ is said to be non-returnable (with respect to f) if $[c]f_*([c]) \cdots f_*^{p-1}([c]) \neq 1 \in \pi_1(F, x_0)$, equivalently $[c] \neq [\bar{\gamma} * (f \circ \gamma)]$ in $\pi_1(F, x_0)$ for any path γ from x_i to x_0, where x_i is a fixed point of f.

It is known that the period p of an irreducible, periodic automorphism satisfies $p \geq 2g + 2$ ([19], see also [33] for related results). Hence to prove Corollary 1.9, we need only the case where $p \geq 6$.

Let us suppose that f has period $p > 2$. Theorem 1.8 follows from:

Proposition 5.1 *For a given periodic automorphism $f : F \to F$ of period $p > 2$ with $f(x_0) = x_0$, there exists a constant C_p depending on the period p of f such that if a geodesic closed path c (with base point x_0) has the length $l_c \geq C_p$, then $[c]$ or $[\bar{c}]$ is non-returnable with respect to f.*

Proof. Choose an $\langle f \rangle$-invariant hyperbolic metric on F ([26, Section 2]). Then the differential df_{x_0} acts on the tangent space $T_{x_0}F$ as a rotation of angle θ_f $(0 < \theta_f < 2\pi)$; $\theta_f = \frac{2\pi q}{p}$ for some integer q $(0 < q < p)$. Remark that $\theta_f \neq \pi$ by the assumption that $p > 2$.

We set a positive constant Θ_f $(0 < \Theta_f < \pi)$ depending on θ_f as follows.

$$\Theta_f = \begin{cases} \pi - \theta_f & 0 < \theta_f \leq \pi/2 \\ \theta_f & \pi/2 < \theta_f < \pi \\ 2\pi - \theta_f & \pi < \theta_f \leq 3\pi/2 \\ \theta_f - \pi & 3\pi/2 < \theta_f < 2\pi \end{cases}$$

Let $\theta_{f,c}$ denote the angle from $\dot{c}(1)$ to $df_{x_0}(\dot{c}(0))$ for c, where $-\pi < \theta_{f,c} \leq \pi$ (see Figure 9).

Claim 5.2 *$|\theta_{f,c}|$ or $|\theta_{f,\bar{c}}|$ is less than or equal to Θ_f.*

Proof. Let θ_c (resp. $\theta_{\bar{c}}$) denote the angle from $\dot{c}(0)$ to $\dot{c}(1)$ (resp. the angle from $\dot{\bar{c}}(0)$ to $\dot{\bar{c}}(1)$), where $0 \leq \theta_c, \theta_{\bar{c}} < 2\pi$, see Figure 9.

First we consider the case $\theta_c = 0$. If $0 < \theta_f < \pi$, then $|\theta_{f,c}| = |\theta_f| \leq \Theta_f$ holds. If $\pi < \theta_f \leq 3\pi/2$, then $|\theta_{f,c}| = 2\pi - \theta_f = \Theta_f$ holds. If $3\pi/2 < \theta_f < 2\pi$, then $|\theta_{f,c}| = 2\pi - \theta_f < \theta_f - \pi = \Theta_f$ holds. Hence, in all cases, we have the inequality $|\theta_{f,c}| \leq \Theta_f$.

Next suppose that $\theta_c \neq 0$. In this case, we have the following table on the values of θ_f, θ_c, $\theta_{\bar{c}}$, $|\theta_f - \theta_c|$ and $|\theta_f - \theta_{\bar{c}}|$, see Figure 9. Here, we remark that $\theta_{\bar{c}} = 2\pi - \theta_c$ if $\theta_c \neq 0$.

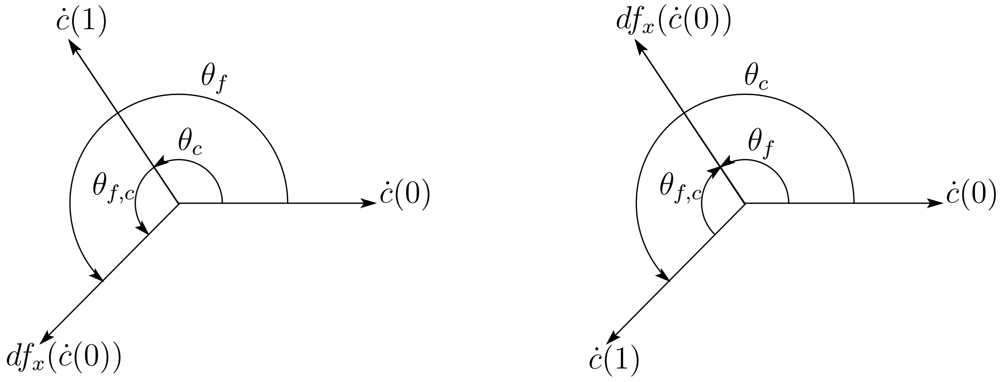

Figure 9: configurations of vectors $\dot{c}(0)$, $\dot{c}(1)$, $df_{x_0}(\dot{c}(0))$ on $T_{x_0}F$

θ_f	θ_c or $\theta_{\bar{c}}$	$\|\theta_f - \theta_c\|$ or $\|\theta_f - \theta_{\bar{c}}\|$
$0 < \theta_f \leq \pi/2$	$0 < \theta_c \leq \pi$	$\|\theta_f - \theta_c\| \leq \pi - \theta_f = \Theta_f$
	$\pi < \theta_c < 2\pi$ $(0 < \theta_{\bar{c}} \leq \pi)$	$\|\theta_f - \theta_{\bar{c}}\| \leq \pi - \theta_f = \Theta_f$
$\pi/2 < \theta_f < \pi$	$0 < \theta_c \leq \pi$	$\|\theta_f - \theta_c\| < \theta_f = \Theta_f$
	$\pi < \theta_c < 2\pi$ $(0 < \theta_{\bar{c}} \leq \pi)$	$\|\theta_f - \theta_{\bar{c}}\| < \theta_f = \Theta_f$
$\pi < \theta_f \leq 3\pi/2$	$0 < \theta_c \leq \pi(\pi < \theta_{\bar{c}} < 2\pi)$	$\|\theta_f - \theta_{\bar{c}}\| \leq 2\pi - \theta_f = \Theta_f$
	$\pi < \theta_c < 2\pi$	$\|\theta_f - \theta_c\| < 2\pi - \theta_f = \Theta_f$
$3\pi/2 < \theta_f < 2\pi$	$0 < \theta_c \leq \pi(\pi < \theta_{\bar{c}} < 2\pi)$	$\|\theta_f - \theta_{\bar{c}}\| \leq \theta_f - \pi = \Theta_f$
	$\pi < \theta_c < 2\pi$	$\|\theta_f - \theta_c\| \leq \theta_f - \pi = \Theta_f$

In all cases, we have $\min\{|\theta_f - \theta_c|, |\theta_f - \theta_{\bar{c}}|\} \leq \Theta_f$.

Moreover, if $0 \leq |\theta_f - \theta_c| \leq \pi$ (resp. $\pi < |\theta_f - \theta_c| < 2\pi$) holds, then $|\theta_{f,c}| = |\theta_f - \theta_c|$ (resp. $|\theta_{f,c}| \leq \pi < |\theta_f - \theta_c|$) holds. Hence, in either case, $|\theta_{f,c}| \leq |\theta_f - \theta_c|$ holds. We have also the inequality $|\theta_{f,\bar{c}}| \leq |\theta_f - \theta_{\bar{c}}|$ in the same way.

Consequently, we conclude that $\min\{|\theta_{f,c}|,\ |\theta_{f,\bar{c}}|\} \leq \Theta_f$ holds. $\square$

Denote by l_c the length of the geodesic closed path c.

Claim 5.3 *Suppose that $|\theta_{f,c}| \le \Theta_f$ holds. If we have*

$$\cosh \frac{l_c}{2} \ge \frac{1}{\sin \frac{\pi}{2p}},$$

then the homotopy class $[c][f \circ c] \cdots [f^{p-1} \circ c]$ is nontrivial in $\pi_1(F, x_0)$.

Proof. Since $\theta_f = \frac{2\pi q}{p}$, by definition of Θ_f, a simple computation shows that $\frac{\pi}{p} \le \pi - \Theta_f < \pi$, which implies $\sin \frac{\pi}{2p} \le \sin \frac{\pi - \Theta_f}{2}$. Thus we have

$$\cosh \frac{\ell_c}{2} \ge \frac{1}{\sin \frac{\pi}{2p}} \ge \frac{1}{\sin \frac{\pi - \Theta_f}{2}} \ge \frac{1}{\sin \frac{\pi - |\theta_{f,c}|}{2}}.$$

We regard the hyperbolic plane $\mathbb{H}^2$ as the universal cover of F. Let $\rho : \pi_1(F, x_0) \to \mathrm{Isom}^+(\mathbb{H}^2)$ be the holonomy representation. We fix a lift $\widetilde{x_0} \in \mathbb{H}^2$ of x_0. Let $\widetilde{c} \subset \mathbb{H}^2$ be the lift of c such that $\widetilde{c}(0) = \widetilde{x_0}$. Let $\widetilde{f \circ c} \subset \mathbb{H}^2$ be the lift of $f \circ c$ such that $\widetilde{f \circ c}(0) = \widetilde{c}(1)$. Successively we have the lifts $\widetilde{f^2 \circ c}, \ldots, \widetilde{f^{p-1} \circ c}$, see Figure 10. Note that since f is an isometry, the lifts $\widetilde{c}, \widetilde{f \circ c}, \ldots, \widetilde{f^{p-1} \circ c}$ have the same length l_c.

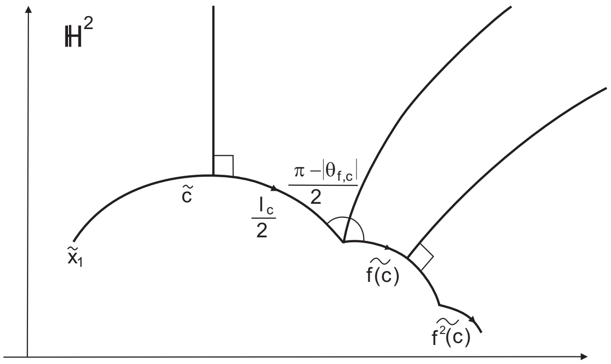

Figure 10: lifts of geodesic paths

Then we have

$$\widetilde{f^{p-1} \circ c}(1) = \rho([c])\rho([f \circ c]) \cdots \rho([f^{p-1} \circ c])(\widetilde{x_0}).$$

By the hyperbolic cosine law for an infinite hyperbolic triangle with exactly one ideal vertex [25, Theorem 3.5.6], the assumption

$$\cosh \frac{l_c}{2} \ge \frac{1}{\sin \frac{\pi - |\theta_{f,c}|}{2}}$$

guarantees that the perpendicular bisectors of $\widetilde{c}$, $\widetilde{f \circ c}$, $\widetilde{f^2 \circ c}, \dots, \widetilde{f^{p-1} \circ c}$ are all mutually disjoint. This implies that

$$\rho([c])\rho([f \circ c]) \cdots \rho([f^{p-1} \circ c])(\widetilde{x_0}) \neq \widetilde{x_0} \ .$$

Thus $\rho([c][f \circ c] \cdots [f^{p-1} \circ c])$ is nontrivial. Since ρ is faithful, $[c]f_*[c] \cdots f_*^{p-1}[c] \neq 1$ in $\pi_1(F, x_0)$. $\square$

Let us put $C_p = 2\cosh^{-1}(\frac{1}{\sin\frac{\pi}{2p}})$. Assume that $\ell_c \geq C_p$. If $|\theta_{f,c}| \leq \Theta_f$, then l_c satisfies the inequality in Claim 5.3 and $[c]$ is non-returnable with respect to f. If $|\theta_{f,c}| > \Theta_f$, then by Claim 5.2, $|\theta_{f,\bar{c}}| \leq \Theta_f$ holds. Since $\ell_{\bar{c}} = \ell_c$, applying the same argument to $\bar{c}$ instead of c, we can show that $[\bar{c}]$ is non-returnable with respect to f. $\square$

We conclude this section with the following remark.

Remark. Assume that $\mathrm{Fix}(f) = \{x_0\}$ and $|A_f - I| = \pm 1$, where A_f is a presentation matrix of the isomorphism $f_\sharp : H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$ induced from f. Note that a mapping torus obtained by a longitudinal surgery on a torus knot (given in Subsection 2.1) satisfies this condition, see [5, p.73]. Since $|A_f - I| = \pm 1$, $\Phi' : H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$ defined by $\Phi'(w) = f_\sharp(w) - w$ for $w \in H_1(F; \mathbb{Z})$ is surjective. Thus every element in $H_1(F; \mathbb{Z})$ is written as $-\alpha + f_\sharp(\alpha)$ for some $\alpha \in H_1(F; \mathbb{Z})$. This suggests that it is not easy to detect non-returnable elements in $\pi_1(F, x_0)$ in general.

6 Applications to surface-automorphisms

In this final section, we translate Theorems 1.6, 1.10 and Corollary 1.9 into terminologies of surface-automorphisms.

Let F be a closed, orientable surface of genus ≥ 2 and f an automorphism of F with $f(x_0) = x_0$ for some point x_0. (Recall that the condition "$f(x_0) = x_0$" implies $f(D_0) = D_0$ for some small diskal neighborhood D_0 of x_0.) If f' is an automorphisms of F isotopic to f which satisfies $f'(x_0) = x_0$. Isotoping f' to f, we obtain a closed curve c base at x_0 which traces x_0 under the isotopy. We call c the *sliding curve* of f' and write $f' = f_c$. Let f'_1 and f'_2 be automorphisms isotopic to f such that their sliding curves represent the same element of $\pi_1(F, x_0)$. Then following Birman [1, Chapter 4], there is an isotopy between them keeping x_0 invariant. In the following, we denote $F - \mathrm{int}D_0$ by $\hat{F}$.

Corollary 6.1 *Let F and f be as in Theorem 1.6. Then the following three conditions are equivalent.*

(1) *The restriction $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.*

(2) $[c]f_*([c])\cdots f_*^{p-1}([c]) \neq 1 \in \pi_1(F, x_0)$.

(3) $[c] \neq [\bar{\gamma} * (f \circ \gamma)]$ *in $\pi_1(F, x_0)$ for any path γ from x_i to x_0, where x_i is a fixed point of f.*

Corollary 6.2 *Let F, f and C_p be as in Theorem 1.6. Then if $L([c]) > C_p$, then the restriction $f_c|_{\hat{F}}$ or $f_{\bar{c}}|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.*

As we mentioned just after Corollary 1.9, considering the angle from $\dot{\gamma_{[c]}}(0)$ to $\dot{\gamma_{[c]}}(1)$, we can detect $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism or $f_{\bar{c}}|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.

Corollary 6.3 *Let F and f be as in Theorem 1.10. Then the following two conditions are equivalent.*

(1) *The restriction $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.*

(2) $[c]f_*([c])\cdots f_*^{p-1}([c]) \in \pi_1(F, x_0)$ *is filling.*

Proof of Corollaries 6.1, 6.2 and 6.3. Consider the mapping torus $(F \times [0,1])/\{(x,0) = (f_c(x),1)\}$ with the gluing map f_c. Since f_c is isotopic to f on F, we can apply a level ($t \in [0,1]$) preserving isotopy to $F \times [0,1]$ so that the gluing map becomes f. This level preserving isotopy deforms the vertical segment $\{x_0\} \times [0,1]$ to a monotone arc t whose projection is c; t gives a section s in M_f by identifying its endpoints. Note that $(\hat{F} \times [0,1])/\{(x,0) = (f_c|_{\hat{F}}(x),1)\} \cong M_{f_c} - \mathrm{int}N(s_0) \cong M_f - \mathrm{int}N(s)$. Then following [32], [24], s_c is hyperbolic if and only if the monodromy map $f_c|_{\hat{F}}$ is isotopic to a pseudo-Anosov automorphism.

Hence Corollaries 6.1, 6.2 and 6.3 follow from Theorems 1.6, 1.9 and 1.10. □

We conclude this paper by stating the corresponding result for the case where the genus of F is less than two. In such situations, $f_c|_{\hat{F}}$ is isotopic to $f|_{\hat{F}}$ on $\hat{F}$ for any curve c. Thus we have:

Addendum 6.4 *Let $f : F \to F$ be a periodic automorphism of a 2-sphere or a torus with a fixed point x_0. Then for any section s_c in M_f, $M_f - \mathrm{int}N(s_c)$ is Seifert fibered. In other words, $\hat{f}_c$ is isotopic to a periodic automorphism for any curve c.*

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